Numerical solution of 2D unsteady integral boundary layer equations with a discontinuous Galerkin method

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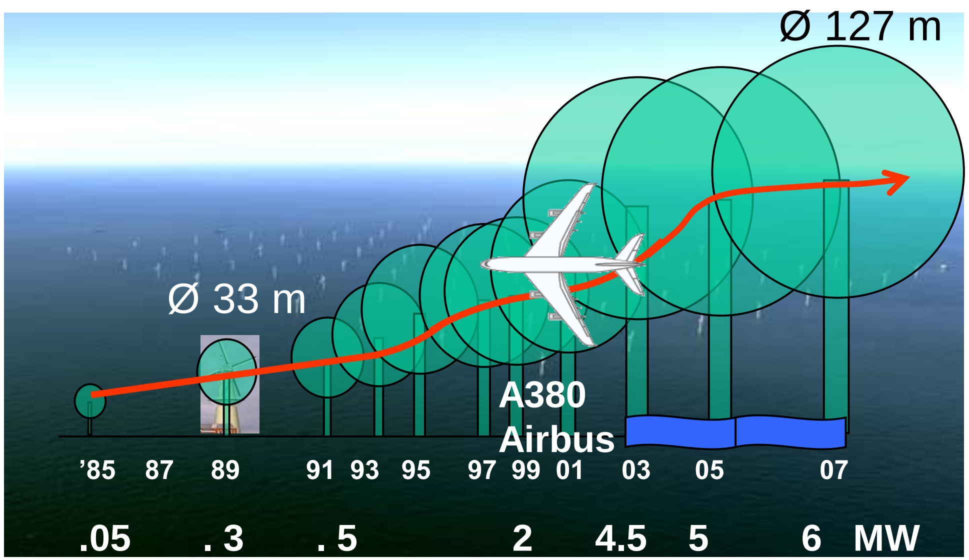
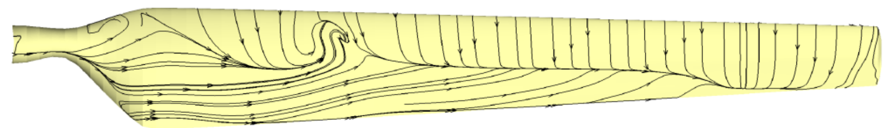
Hüseyin Özdemir

Introduction: motivation

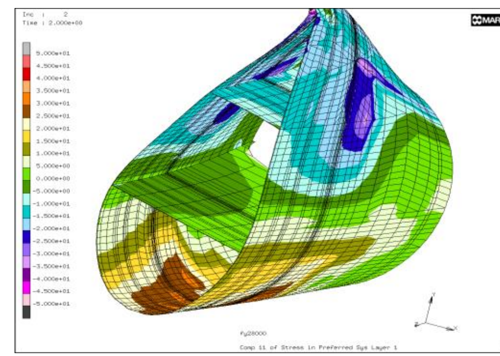


Van Kuik, 2007

- Detailed wind turbine dynamics simulation

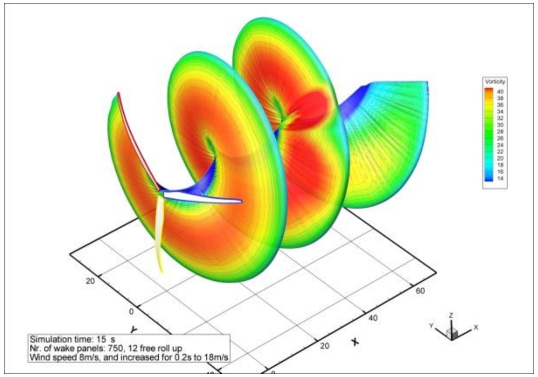


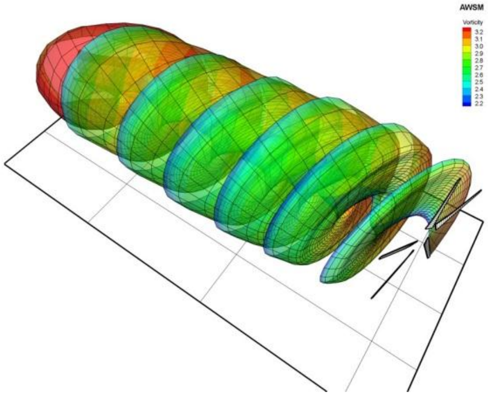
- Local aerodynamic forces, structural stresses and deformations

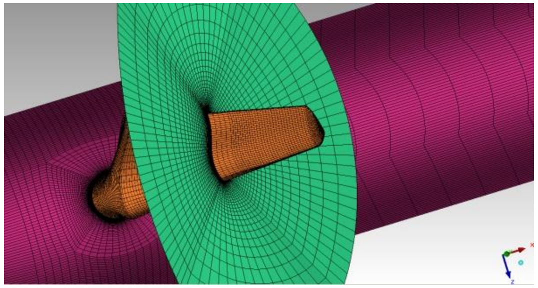


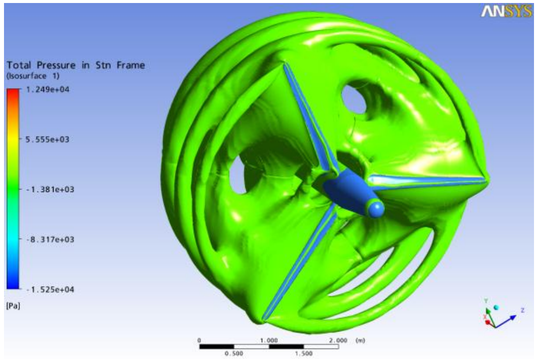
Introduction: available tools

- Engineering tools: ***not accurate enough***
  - 3D, steady state methods: Blade Element, Momentum (BEM), Vortex line method (AWSM),
  - 2D, steady state methods: XFOIL, RFOIL
- CFD tools (CFX): ***too expensive, too much time***
  - Axial-symmetric (1/3rd of the domain)
  - 2.7 M elements
  - ~2 weeks on 16 node cluster









Introduction: approach

***Integral boundary layer method (IBL)* + Panel method + Strong quasi-simultaneous viscous – inviscid interaction**

Navier-Stokes equations

↓

$$\frac{\partial p}{\partial y} = 0$$

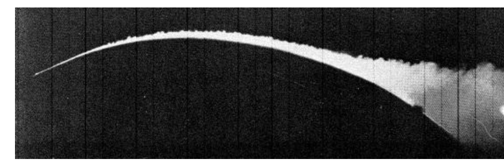
Boundary layer equations

↓

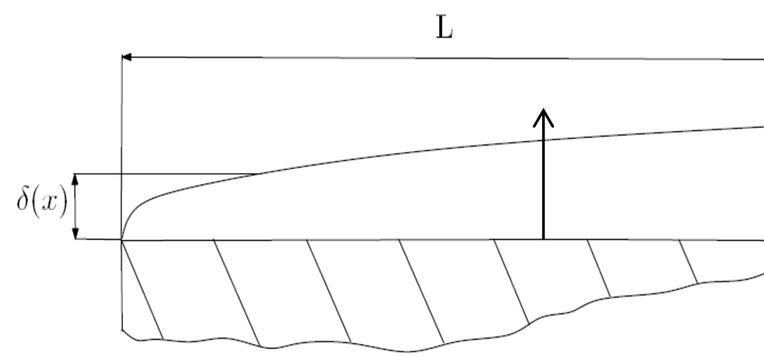
$$\int_0^{y_\infty} |\text{Cont. eqn.}| \times \left(u^{n+1} - u_e^{n+1}\right) + |\text{Mom. eqn.}| \times (n+1)\, u^n dy$$

↓

Integral boundary layer equations



Van Dyke, An Album of Fluid Motion, 11th ed., Parabolic Press, 2007, USA



Governing equations

2D, unsteady integral boundary layer eqns:

$$\frac{\partial \mathbf{F}(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial \mathbf{x}} = \mathbf{S}(\mathbf{u})$$

$$\mathbf{F}(\mathbf{u}) = \begin{bmatrix} \delta^* \\ \delta^* + \theta \\ \frac{C_\tau}{U_S} \end{bmatrix}, \qquad \mathbf{G}(\mathbf{u}) = \begin{bmatrix} u_e \theta \\ u_e \delta^k \\ u_e C_\tau \end{bmatrix},$$

$$\mathbf{S}(\mathbf{u}) = \begin{bmatrix} \frac{C_f}{2} u_e - (\delta^* + \theta)\frac{\partial u_e}{\partial x} - \delta^* \frac{1}{u_e}\frac{\partial u_e}{\partial t} \\ C_D u_e - 2\delta^k \frac{\partial u_e}{\partial x} - 2\theta \frac{1}{u_e}\frac{\partial u_e}{\partial t} \\ C_\tau u_e - C_\tau \frac{\partial u_e}{\partial x} - \frac{2}{U_S}\frac{1}{u_e}\frac{\partial u_e}{\partial t} \end{bmatrix},$$

Closure set:

i.e.:

Shape factors

$$H = \frac{\delta^*}{\theta}, \quad H^* = \frac{\delta^k}{\theta},$$

Friction coefficient

$$C_f(H, \theta),$$

Viscous diffusion coefficient

$$C_D(H, \theta, H^*(H)),$$

Slip velocity

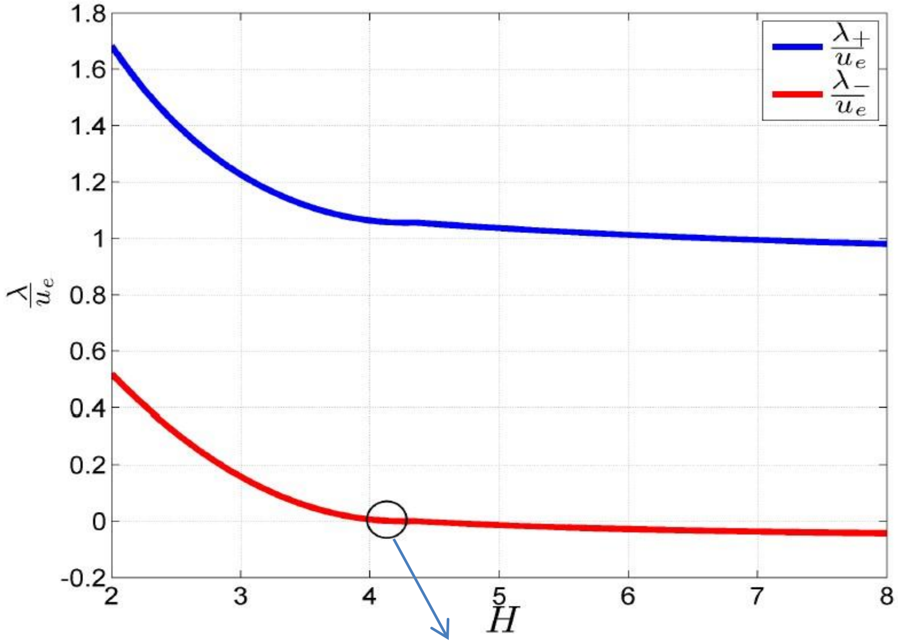
$$U_S(H, H^*(H)),$$

Shear stress coefficient

$$C_\tau(H, H^*(H), U_S).$$

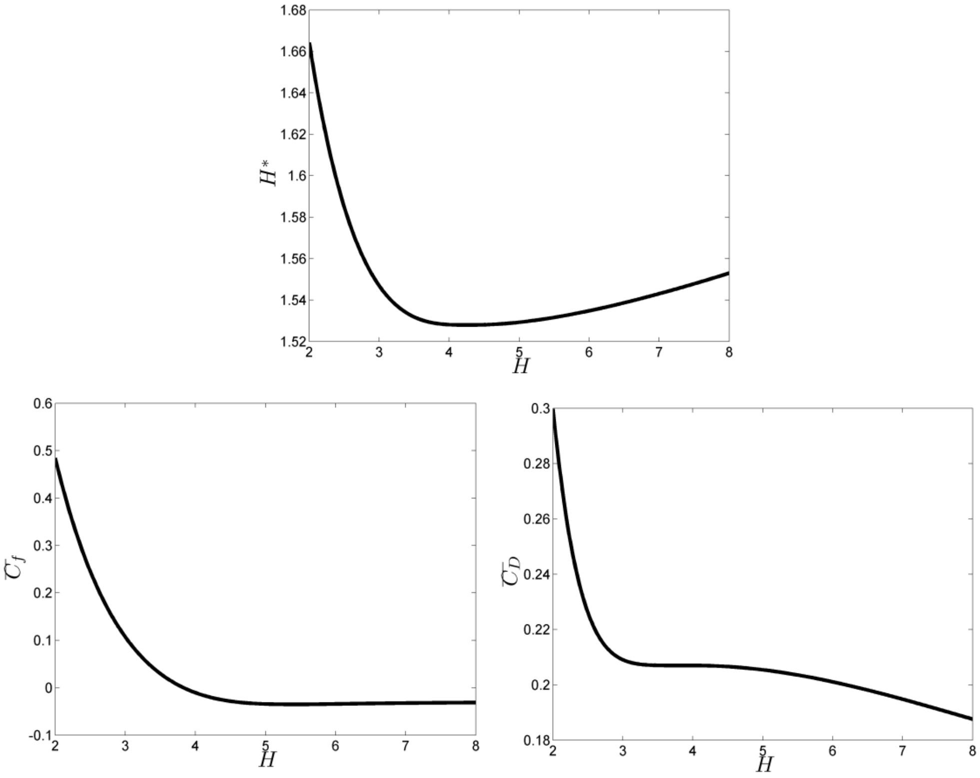
Governing equations: some analysis

Eigenvalues λ− and λ+ for laminar closure relations: system is hyperbolic

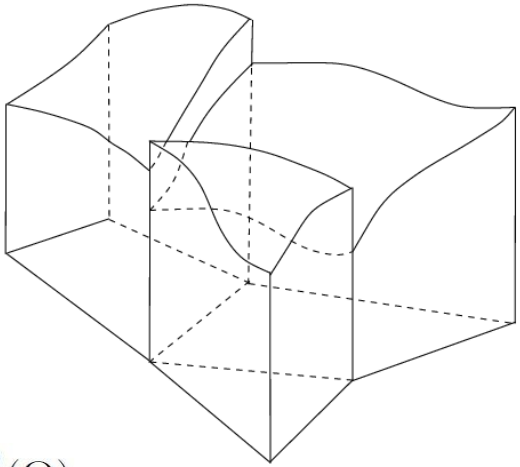


Separation point
λ becomes negative

Closure relations for the *H* dependent variables



Numerical method: Discontinuous Galerkin (DG) Method



Solution vector:

$$\mathbf{u}(\cdot,t) \in U^d, \qquad U \equiv L^2(\Omega),$$

Weak formulation

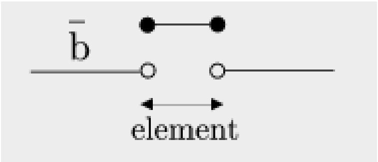
$$(L(\mathbf{u}(\cdot,t)),\mathbf{v}) = (\mathbf{s},\mathbf{v})$$

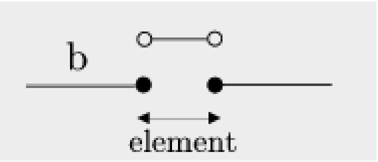
Approximate solution

$$\mathbf{u}_h(\cdot,t) \in U_h^d, \quad U_h = span\{b_{jk}\} \subset U, \, \forall \mathbf{v} \in U^d,$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{jk}(t) b_{jk}(\mathbf{x}), \quad \mathbf{u}_{jk}(t) \in L^2(I_t), \, b_{jk} \in L^2(\Omega),$$

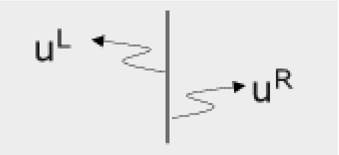
Discrete equation





$$\int_{\Omega_j} L(\mathbf{u}_h(\mathbf{x},t)) b_{jm} d\Omega = \int_{\Omega_j} \mathbf{s} b_{jm} d\Omega,$$

$$\int_{\Omega_j} \frac{\partial \mathbf{u}_{jk}}{\partial t} b_{jk} b_{jm} d\Omega - \int_{\Omega_j} \mathbf{f}_i \frac{\partial b_j m}{\partial x_i} d\Omega + \int_{\partial\Omega_j} b_{jm} \mathbf{h}(\bar{\mathbf{u}}_j, \bar{\mathbf{u}}_l, \mathbf{n}_j) dS = \int_{\Omega_j} \mathbf{s} b_{jm} d\Omega.$$



Numerical method: DG Method

Local Lax-Friedrich flux formula:

$$\mathbf{h}(\bar{\mathbf{u}}_j, \bar{\mathbf{u}}_l, \mathbf{n}_j) = \frac{1}{2}\left\{f(\bar{\mathbf{u}}_j) + f(\bar{\mathbf{u}}_l) - \theta(\bar{\mathbf{u}}_l - \bar{\mathbf{u}}_j)\right\}, \quad \theta = \max_{min(\bar{\mathbf{u}}_j,\bar{\mathbf{u}}_l)\leqslant s\leqslant max(\bar{\mathbf{u}}_j,\bar{\mathbf{u}}_l)} |f'(s)|$$

Explicit multi-stage Runge-Kutta time integration: [Cockburn & Shu]

- Set $\mathbf{U}_k^0 = \mathbf{U}_k^n$;
- For $s = (1, r)$ compute the solution at r intermediate time stages:

$$\mathbf{U}_k^s = \sum_{l=0}^{s-1} \alpha_{sl}\mathbf{W}_k^{sl}, \quad \mathbf{W}_k^{sl} = \mathbf{U}_k^l + \frac{\beta_{sl}}{\alpha_{sl}}\Delta t\mathbf{R}_k^l;$$

- Set $\mathbf{U}_k^{n+1} = \mathbf{U}_k^r$.

Slope limiter: [Cockburn & Shu]

$$|\overline{u_h}|_{TV(0,1)} \equiv \sum_{1\leq j\leq N} |\overline{u}_{j+1} - \overline{u}_j|, \qquad |\overline{u_h}^{s+1}|_{TV(0,1)} \leq |\overline{u_h}^s|_{TV(0,1)}$$

Results

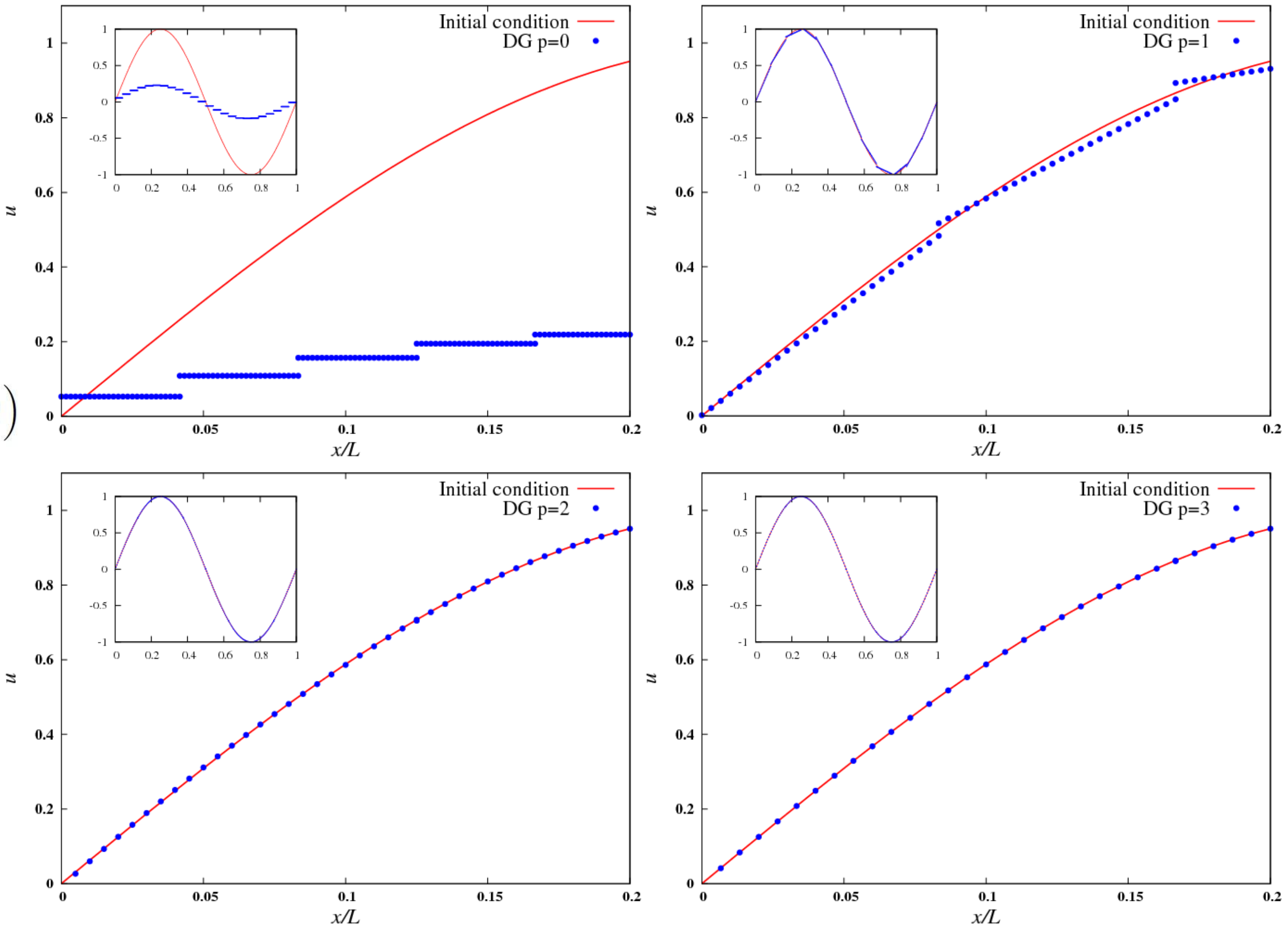
model problem: transport equation, continuous initial condition

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \sin(2\pi x)$$

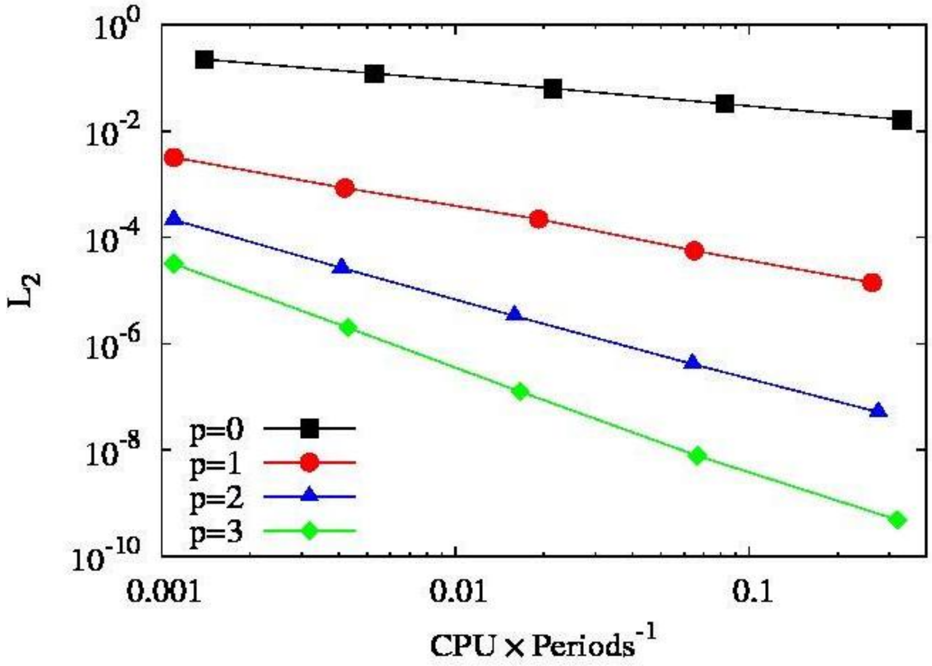
periodic b.c.

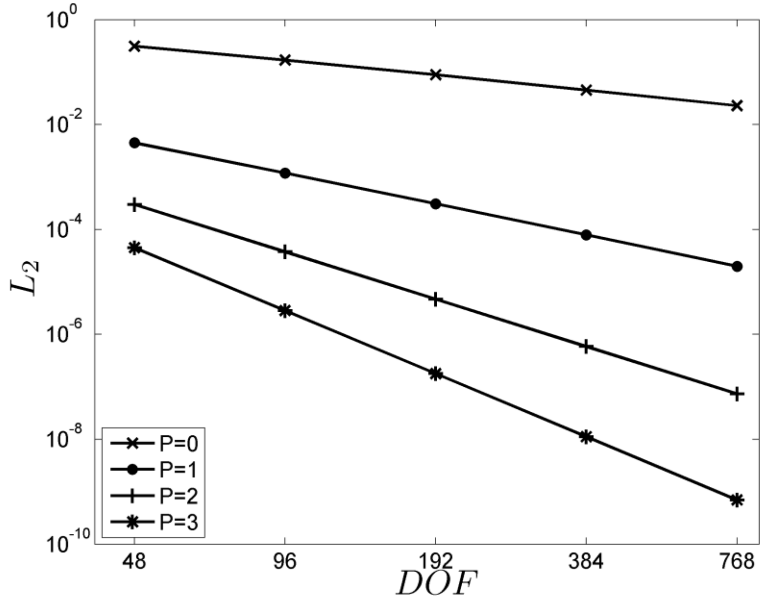
$\Delta x = 1/25$



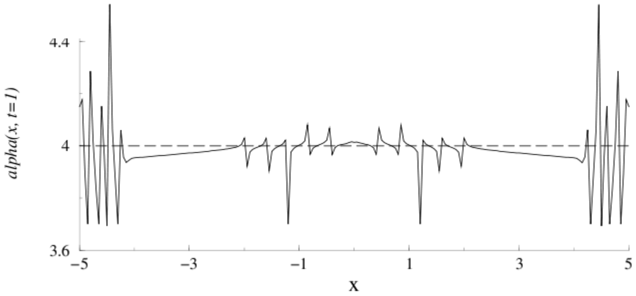
Results

model problem: accuracy





$$|u_{\Delta t}(x,t) - u_{\Delta t_0}(x,t)| = c\Delta t^{\alpha}$$



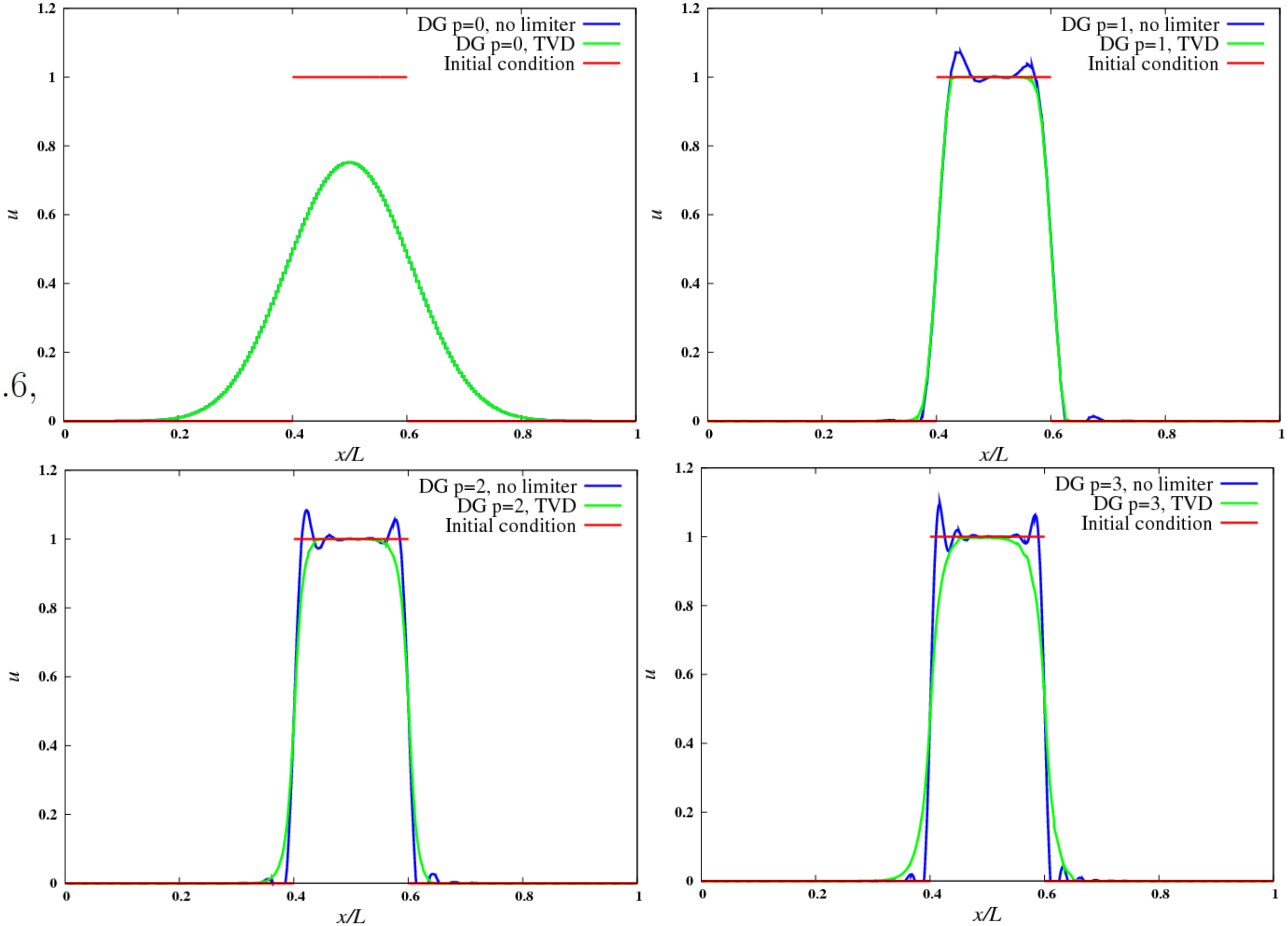
Results

model problem: transport equation, discontinuous initial condition

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \begin{cases} 1, & .4 < x < .6, \\ 0, & \text{otherwise} \end{cases}$$

- periodic b.c.
- slope limiter applied

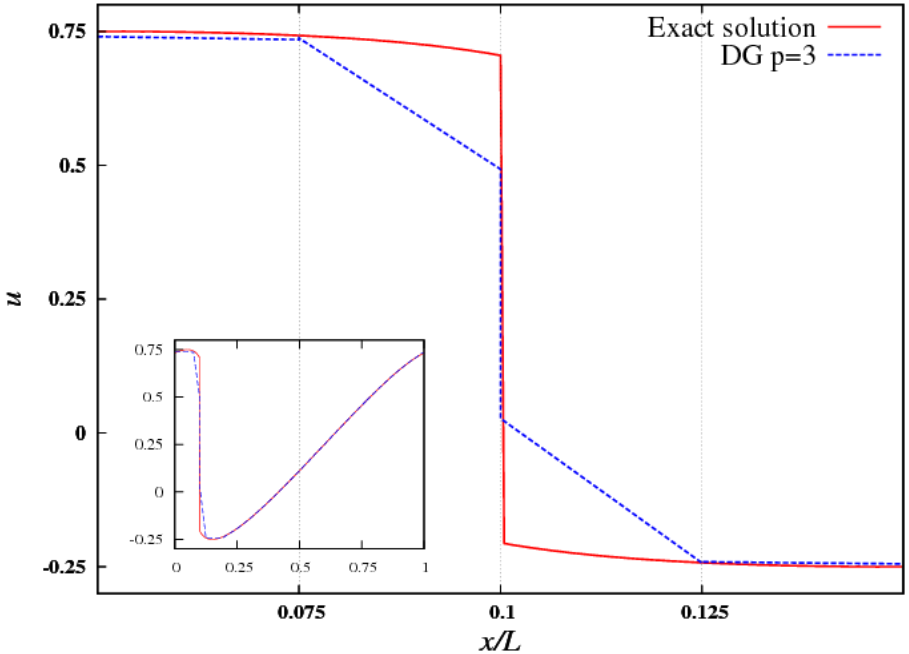


Results

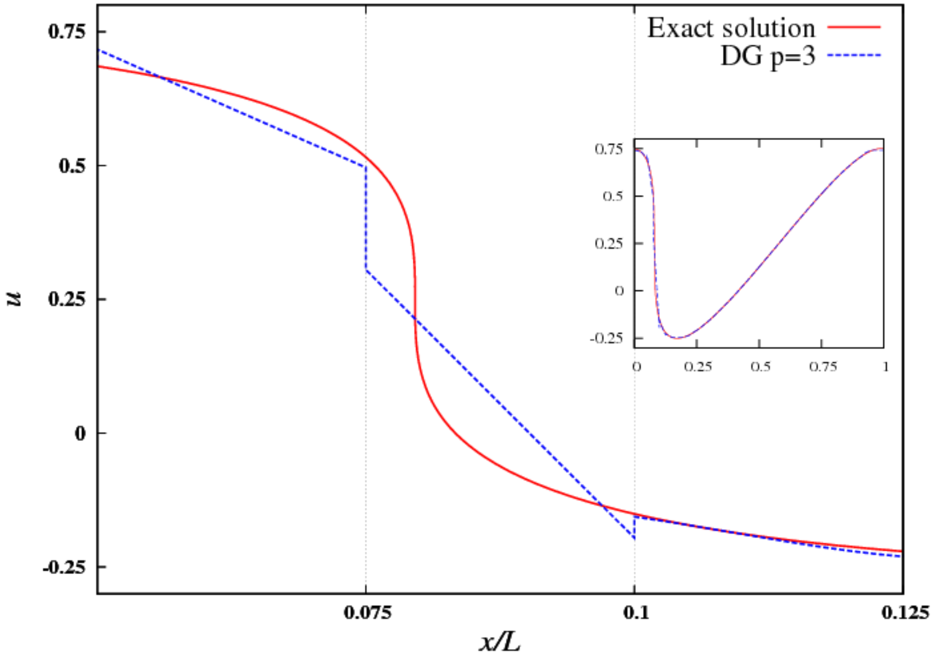
model problem: Burger's equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \qquad u(x,0) = u_0(x) = \frac{1}{4} + \frac{1}{2}\sin(\pi(2x-1)) \qquad \Delta x = 1/160$$

t = 0.4

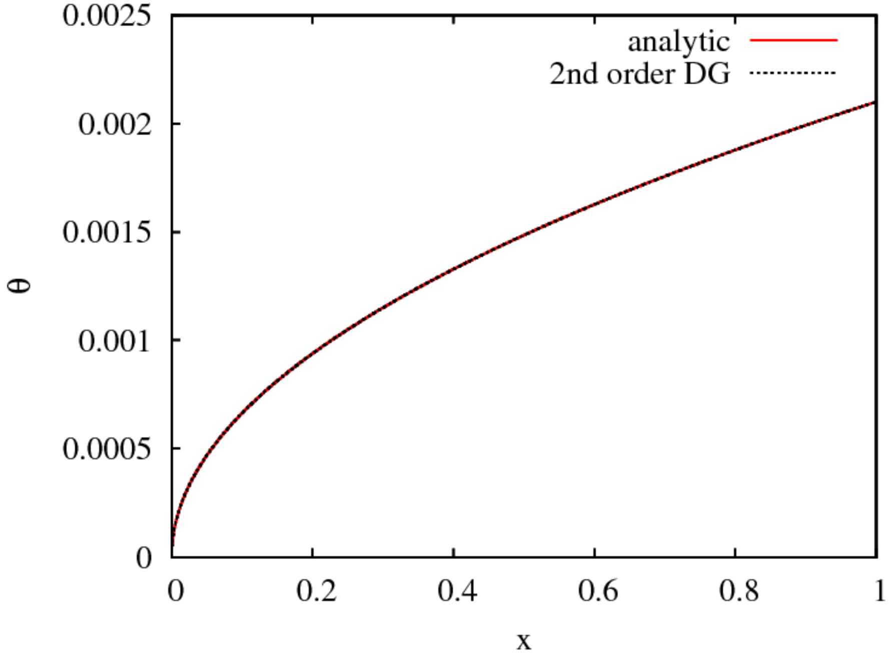


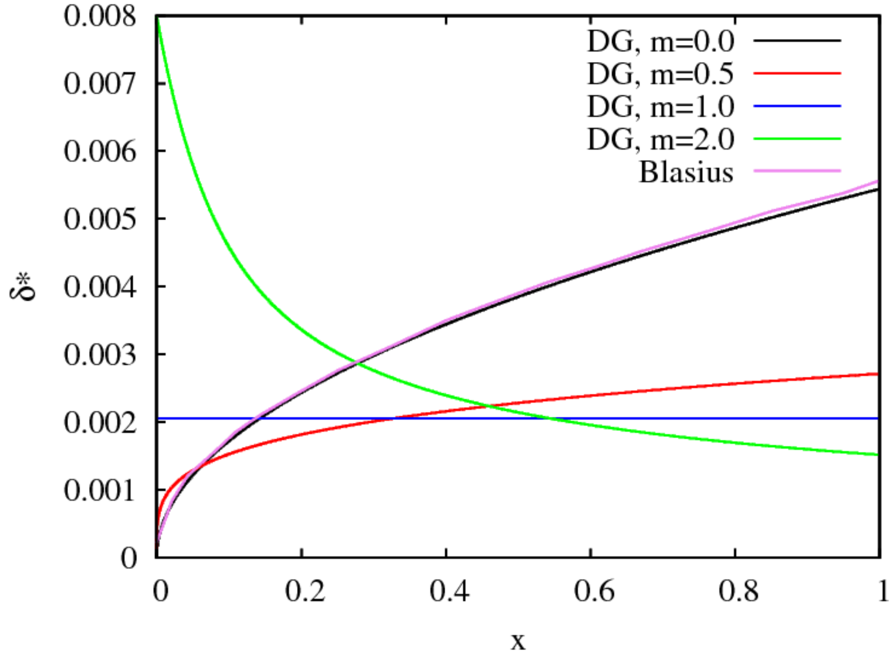
t = 1/π



Results: IBL equations

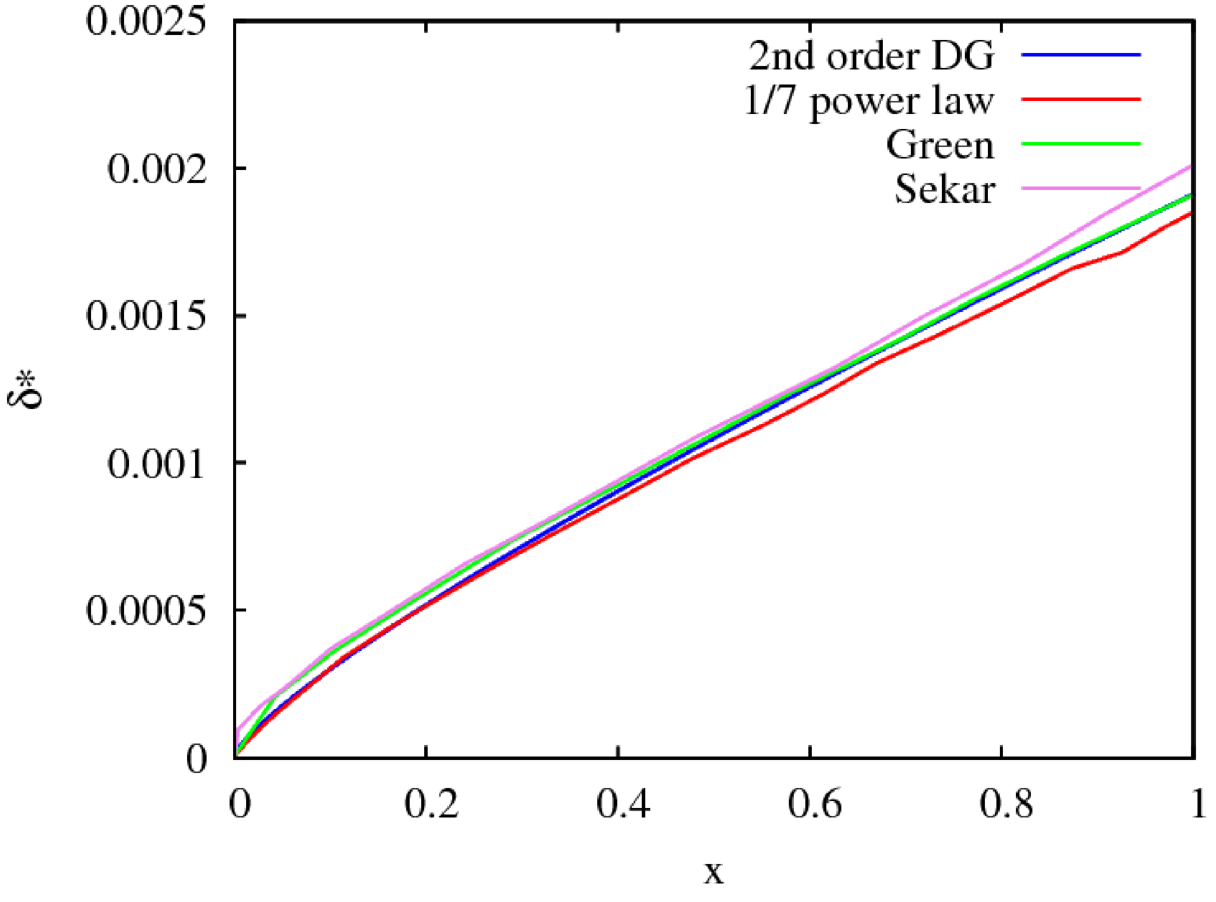
Laminar flow over a flat plate, Re=1e5 $u_{edge} = U_\infty \cdot x^m$





Results: IBL equations

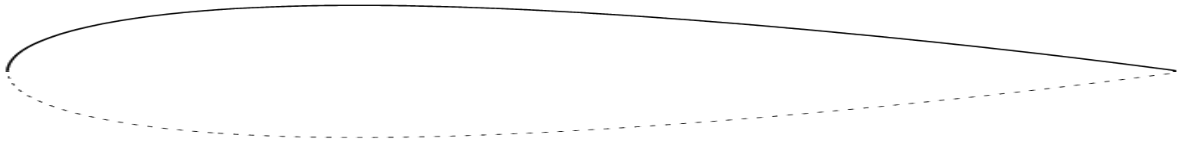
Turbulent flow over a flat plate, Re=1e7



Results: IBL equations

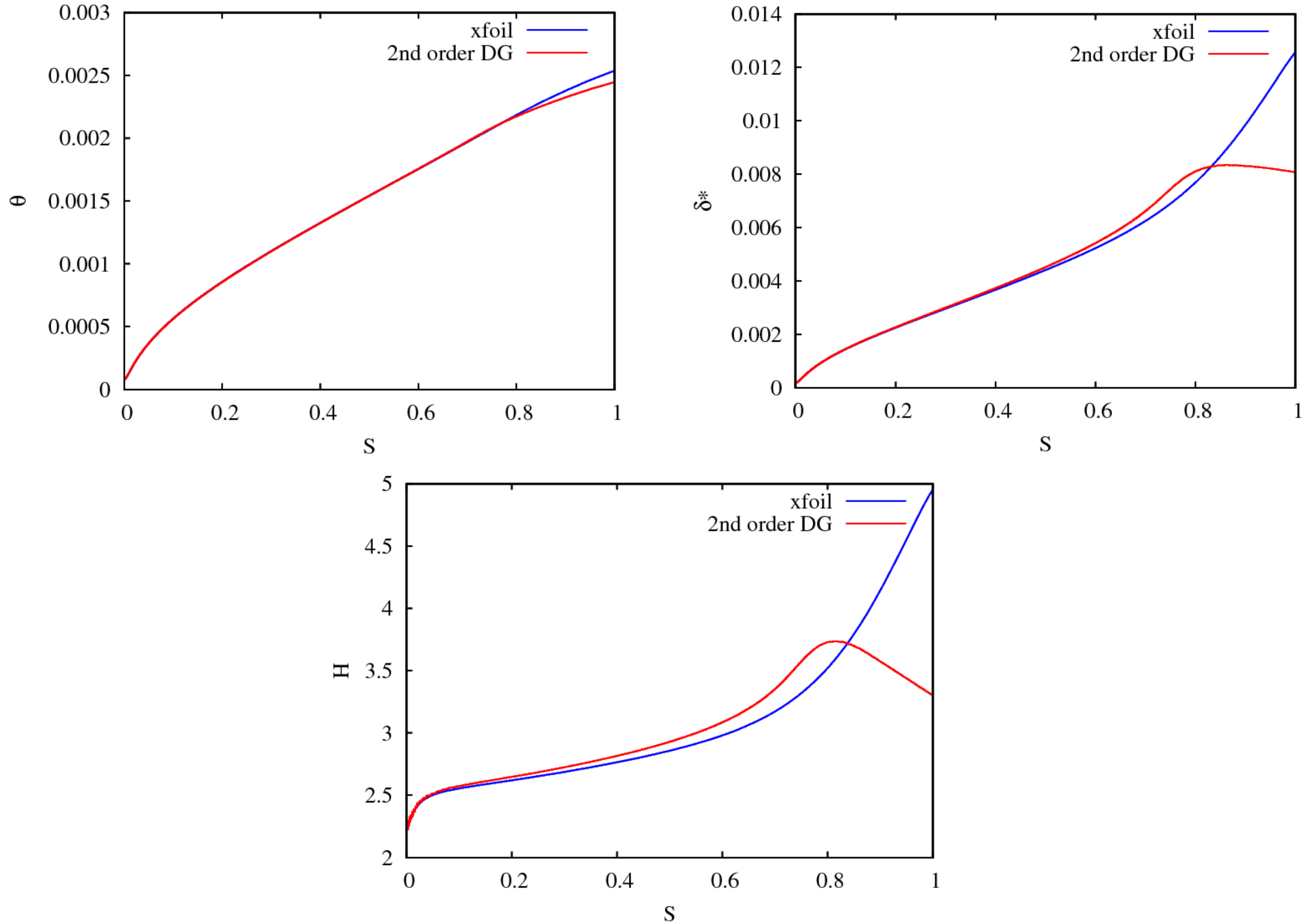
Laminar and turbulent flows over NACA profiles

- Prescribed edge velocity Ue (extracted from XFOIL)
- Dirichlet boundary conditions used
- Initial condition is set
- Only the suction side is considered
- Converging to a steady state problem
- NACA0009 and NACA0012 profiles are used



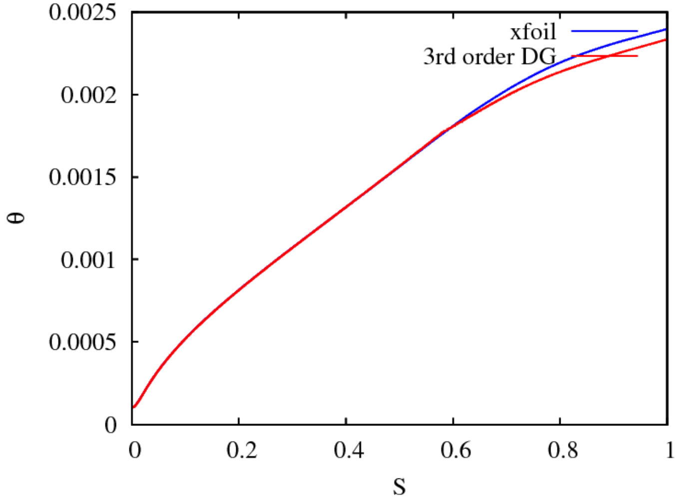
Results: IBL equations

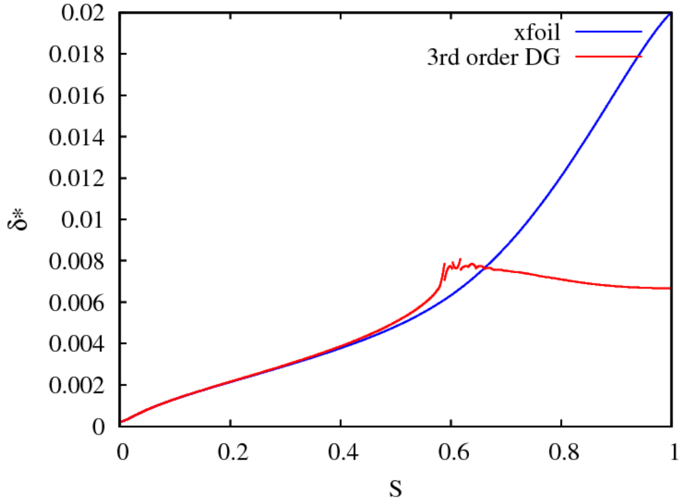
Laminar flow over NACA0009 profile, Re=1e5

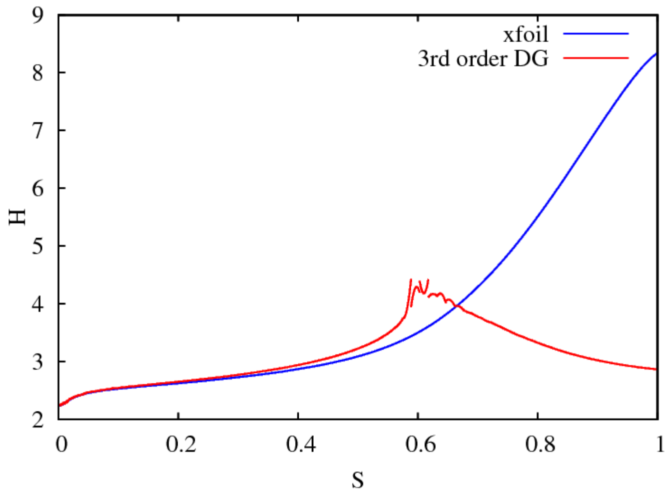


Results: IBL equations

Turbulent flow over NACA0012 profile, Re=1e5



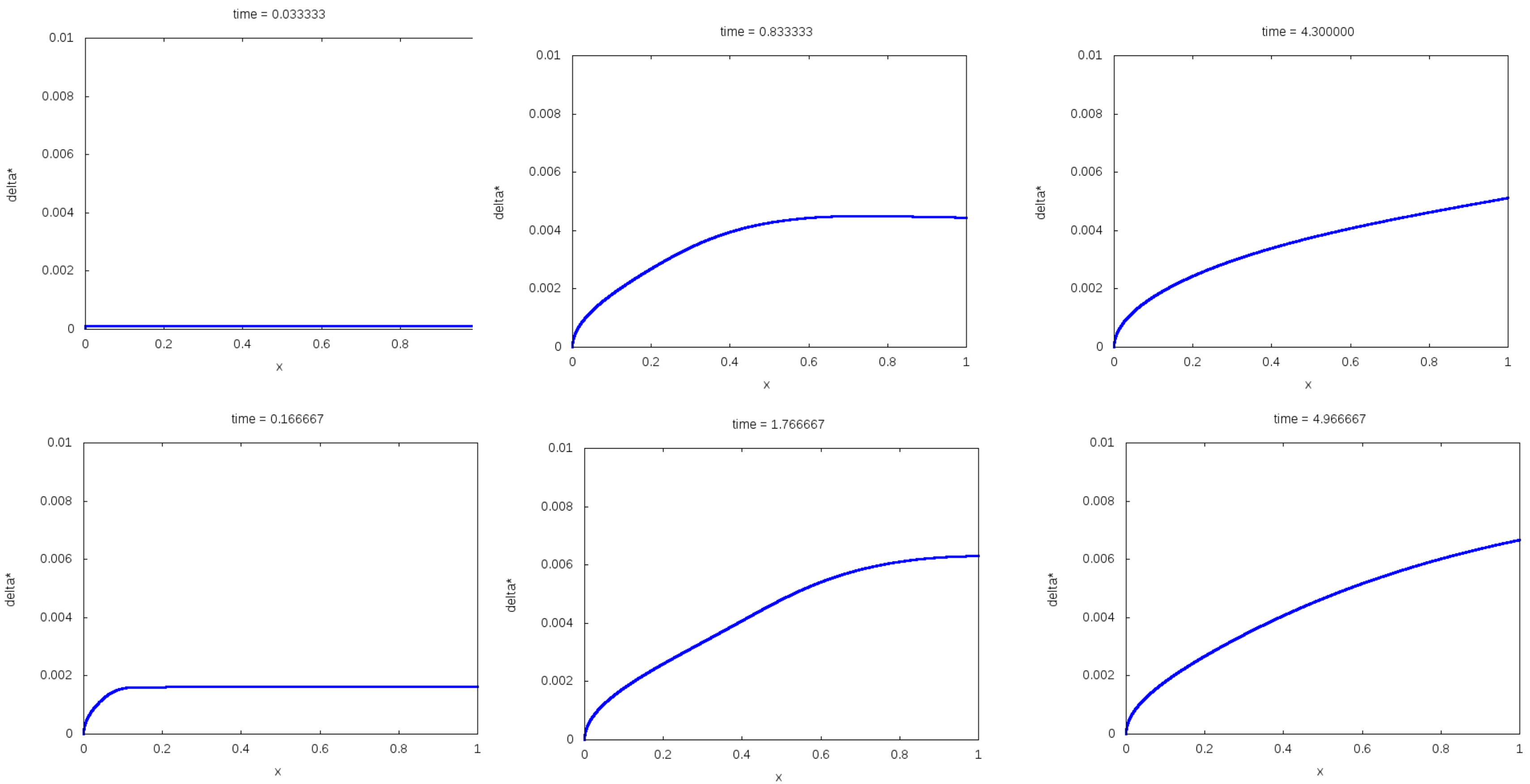




Results: unsteady simulation

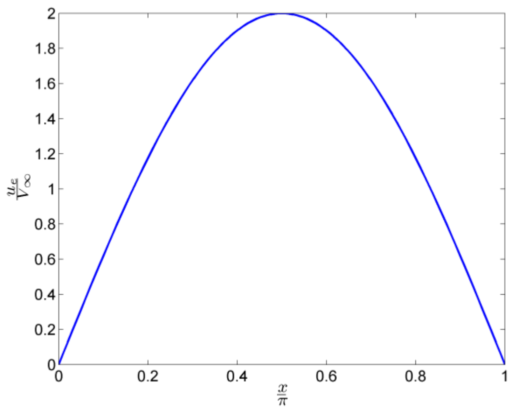
Laminar flow over a flat plate, Re=1e5, with small perturbation in time

$U_e(x,t)$



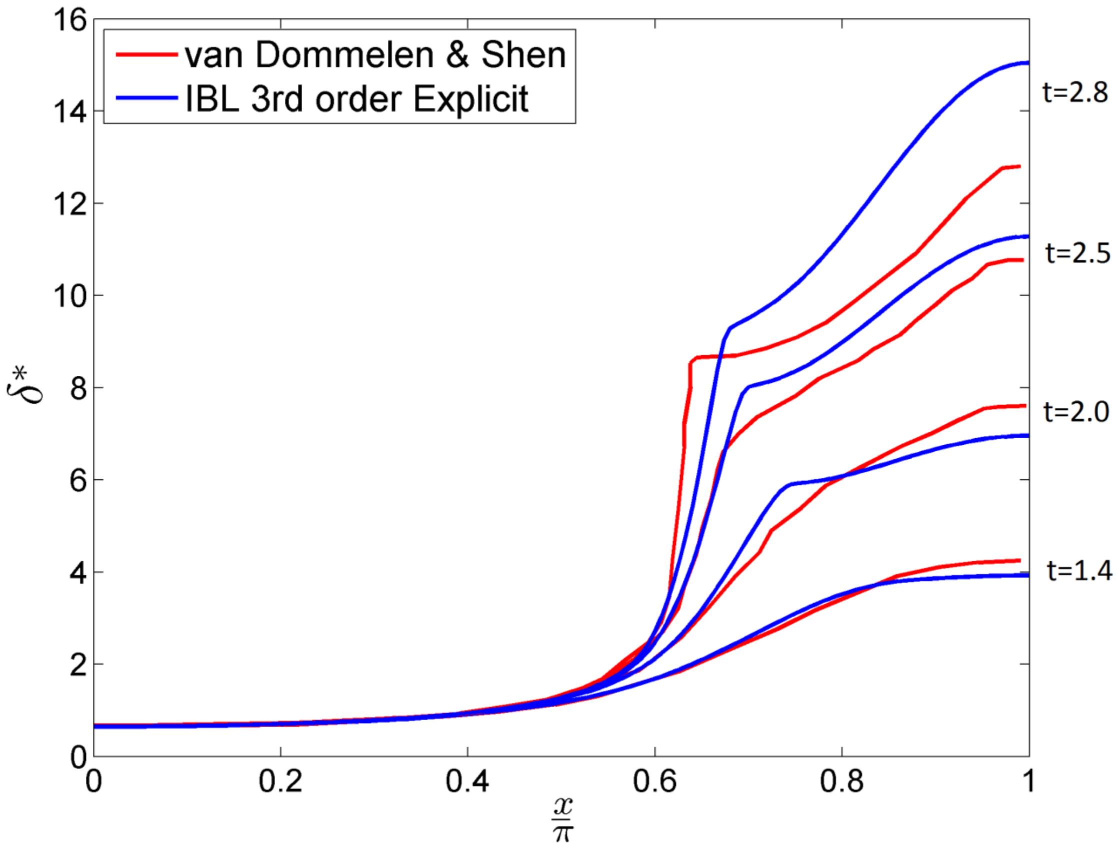
Results: unsteady simulation

Laminar flow over a cylinder



$$u_e(x) = V_\infty 2sin(x)$$

$$\nu = 1, L = \pi$$



Conclusions and Outlook

- Fully laminar and turbulent flows over a flat plate show good agreement with the literature
- Flow over NACA profiles are in good agreement up to separation point

- Non-conservative implementation
- TVBM slope limiter will be implemented
- More experimental data needed for 3D unsteady boundary layers and also for rotational effects

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