Energy research Centre of the Netherlands

Buckling load prediction tools for rotor blades

Model description of tools for buckling of thin-walled beams

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ECN-C--05-103

PREFACE

Within the Dutch research project BLADKNIK buckling load prediction tools were developed for wind turbine rotor blade design. This project can be considered as follow-up development of the European BUCKBLADE project and the Dutch STARION project.

The developments reported here are implemented in the wind turbine design package Focus. This report also serves as reference for the programs and subroutines, and gives a theoretical background of the buckling load prediction methods.

The BLADKNIK project was supported by Senter*Novem*, project No: 2020-02-12-10-007.

ABSTRACT

Within the Dutch research project BLADKNIK buckling load prediction tools were improved and/or developed to be used in wind turbine rotor blade design. These tools include programs that model buckling of a complete cross-section (Finstrip) and programs that model buckling of panels within a cross-section. This report contains descriptions of each of the buckling load prediction methods. For the practical (and fast) engineering 'Design rules' this description is in terms of references to design handbooks and publications. For the panel-based prediction methods this report contains the theoretical background. This includes laminate theory, non-linear strain-displacement relations for curved panels, stability equations derived from energy methods, a description of the solution method, and finally a description of the routines that are developed in the BLADKNIK project. These routines are developed for use in Farob (under Focus), that includes a structural model of a rotor blade. The program Finstrip for buckling of complete cross sections is reported in terms of its functional specifications and the improvements that were issued in the BLADKNIK project. Finally this report contains a functional description of the Farob option to generate input for a Finite Element Package, together with the improvements that were issued in this project.

Keywords

Buckling load, Composite material, Design tools, Rotor blade, Sandwich panel, Wind turbine.

CONTENTS

LIST OF SYMBOLS

A	[N/m]	In-plane or "membrane" stiffness matrix.
$A_{44},\ A_{55}$	[N/m]	Out-of-plane shear stiffnesses of a (sandwich-) panel; for an orthotropic sandwich: $A_{44} = G_{23}\,h^2/t_\mathrm{c}$; $A_{55} = G_{13}\,h^2/t_\mathrm{c}$.
B	[N]	Coupling matrix between in-plane and bending stiffnesses.
b	[m]	Width of the panel c.q. the half-waves or bulges.
C	[m/N]	Inverse of the membrane stiffness matrix: $C = A^{-1}$.
D	[Nm]	Out-of-plane bending stiffness matrix.
$\tilde{D}$	[Nm]	Reduced bending stiffness matrix: $\tilde{D} = D - B \cdot A^{-1} \cdot B$.
$E_1,\ E_2$	[N/m^2]	Young's moduli in the direction of the largest- and smallest principal stiffness respectively.
f	[Nm]	Airy stress function that applies to Donnell's in-plane equilibrium equations (see p.347 of [25]): $N_\mathrm{x} = f_{,\mathrm{yy}}$; $N_\mathrm{xy} = -f_{,\mathrm{xy}}$; $N_\mathrm{y} = f_{,\mathrm{xx}}$.
$frac$		Used as relative fraction of shear loading vs combined shear and compression.
G_{12}	[N/m^2]	In-plane shear modulus.
$G_{13},\ G_{23}$	[N/m^2]	Out-of-plane shear moduli in longitudinal- and transverse direction.
h	[m]	Core thickness, over which out-of-plane shear deformation is modelled.
H		Matrix in the set of equations for sandwich panel buckling, Appendix B.
K	[m]	The matrix product $K = A^{-1} \cdot B$
$k_\mathrm{b},\ k_\mathrm{c},\ k_\mathrm{s}$		Dimensionless buckling factors $k_\mathrm{b} = \lvert N_\mathrm{x\,bend}\rvert/[\,(\pi/b)^2\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\,]$, $k_\mathrm{c} = -\bar{N}_\mathrm{x}/[\,(\pi/b)^2\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\,]$, and $k_\mathrm{s} = \lvert N_\mathrm{xy}\rvert/[\,(\pi/b)^2\sqrt{\tilde{D}_{22}\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}}\,]$.
k_supp	[N/m^3]	Stiffness of an elastic support, see Appendix A and B.
L	[m]	Length of the panel c.q. of the half-waves or bulges.
M_x	[N]	Bending moment per unit width, giving longitudinal curvature.
M_xy	[N]	Twisting moment in the shell panel per unit width.
M_y	[N]	Bending moment per unit length, giving transverse curvature.
N_x	[N/m]	Longitudinal-load distribution per unit width.
N_xy	[N/m]	Shear-load distribution per unit width.
N_y	[N/m]	Load distribution in transverse direction per unit width.
$Q_\mathrm{xz}, Q_\mathrm{yz}$	[N/m]	Out-of-plane panel shear loading per unit width.
$R_\mathrm{b}, R_\mathrm{c}, R_\mathrm{s}, R_\mathrm{t}$		Rel. load levels for **b**ending, ax. **c**ompression, **s**hear, and **t**ransv. compression. defined as e.g. $R_\mathrm{c} = N_\mathrm{x}/N_\mathrm{x.cr}$.
$R_\mathrm{x},\ R_\mathrm{y}$	[m]	Radii of geometric curvature in longitudinal- and transverse direction.
R_y^*	[m]	Effective radius of transverse curvature, including K matrix elements.
R_xy	[m]	Inverse of the geometric twist of a panel.
$r_1\ ,\ r_2$		Stiffness ratios for sandwich panels: $r_1 = (\pi/b)^2\,(\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\,/A_{55})\,(1+\beta)/2$; $r_2 = (\pi/b)^2(\tilde{D}_{22}/A_{44})\,(1+\beta)/2$.
r		Average stiffness ratio: $r = \sqrt{r_1\,r_2}$.
S_{ij}, T_{ij}		Matrices used in solving the 2-point boundary value problem in section A.4.
t	[m]	Wall thickness of the panel.
t_c	[m]	Thickness of the elastic core of a sandwich.
$\bar{U}$		Matrix in the set of first-order stability equations (section A.4).
u	[m]	In-plane displacement in longitudinal direction of a point of the panel.
v	[m]	In-plane displacement in transverse direction of a point of the panel.
w	[m]	Out-of-plane displacement of a point of the panel.
Y		Used in the expression of v.d. Neut for buckling of orthotropic cylinders.
Z		Curvature parameter: $Z = b^2/(R_\mathrm{y}\sqrt{\tilde{D}_{22}\,C_{22}}\,)$; for isotropic panels: $Z = b^2 \cdot \sqrt{12(1-\nu^2)}/(R_\mathrm{y}\,t)$.
$x\ ,\ y$	[m]	Longitudinal- and transverse co-ordinates along the panel.
z	[m]	Out-of-plane co-ordinate of the panel, for a blade positive inward.

α		Dimensionless half-wave length: $\alpha = (L/b)\ \sqrt[4]{\tilde{D}_{22}/\tilde{D}_{11}}$.
β		Orthotropy parameter: $\beta = (\tilde{D}_{12} + 2\,\tilde{D}_{66})/\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}$.
$\beta_{\rm y}$	[rad]	Rotation of the panel cross-section about the *x*-axis: $\beta_{\rm y} = -w_{,y} - v/R_{\rm y} + \gamma_{\rm yz}$.
$\epsilon_{\rm x},\ \epsilon_{\rm y}$		Longitudinal- and transverse strain.
$\gamma_{\rm xy}$	[rad]	Shear strain, positive for reduced angle between *x*-axis and *y*-axis.
$\gamma_{\rm xz}$	[rad]	Out-of-plane shear deformation in longitudinal direction.
$\gamma_{\rm yz}$	[rad]	Out-of-plane shear deformation in transverse direction.
$\kappa_{\rm x}$	[1/m]	Elastic panel curvature about the transverse (*y*-) axis: $\kappa_{\rm x} = -w_{,xx}$.
$\kappa_{\rm y}$	[1/m]	Elastic panel curvature about the longit. (*x*-) axis: e.g. $\kappa_{\rm y} = -w_{,yy} - v_{,y}/R_{\rm y}$.
$\kappa_{\rm xy}$	[1/m]	Elastic twist of the panel, expressed as: $\kappa_{\rm xy} = -2\,w_{,xy} - 2\,v_{,x}/R_{\rm y}$.
λ		Factor on the load distribution for which buckling is calculated.
$\mu_{\rm x}, \mu_{\rm y}$	[1/m]	Longitudinal- and transverse wave-length parameters; $\mu_{\rm x} = \pi/L$ and $\mu_{\rm y} = \pi/b$ with L and b half-wave dimensions.
$\mu_{\rm b}$	[1/m]	Wave-length parameter for the wrinkling mode: $\mu_{\rm b} = \pi/b$.
ν_{12}		Poisson's ratio; contraction in the "*2*" direction due to an elongation in the "*1*" direction.
ν_{21}		Contraction in the "*1*" direction due to an elongation in the "*2*" direction. These Poisson's ratios apply to $\nu_{12}\,E_2 = \nu_{21}\,E_1$.
π		Trigonometric constant: $\pi = 3.1415926536$ (value used in the routines).
ψ		Coefficient in the description of a linear varying longitudinal compression $N_{\rm x,\,smallest\,compression} = \psi\,N_{\rm x,\,largest\,compression}$.
$(.)_{,x}\ (.)_{,y}$	[1/m]	Differentiation with respect to the *x*-coordinate or *y*-coordinate respectively.

TERMINOLOGY

In this document the following terms are used for programs, directions of loads and deformations, file formats, etcetera:

Expression	Description
Crostab	Program (ECN) for cross-sectional analysis, including panel-buckling.
Farob	Program (WMC) for fatigue analysis of a rotor blade, within Focus.
Finstrip	Program (WMC) for linear buckling analysis of a blade cross section.
Focus	Design package (WMC) for wind turbines, having a detailed blade model.
MSC.MARC	Finite-element package with non-linear options for buckling.
Phatas	Program (ECN) running under Focus for wind turbine response in time domain.
`*.buc` file	File format that contains the geometry, material definition, and the layup of a cross section. This was introduced for **buc**kling analyses with Finstrip.
core	Used for a sandwich panel layup. The 'core' is the layer (or some layers) in the middle of a laminated panel that has out-of-plane shear flexibility. Usually the core is of light foam or a light honeycomb structure.
facing	Used for a sandwich panel layup. The 'facing' is the layer (or some layers) on the outer surfaces of a laminated panel that has in-plane load carrying capabilities, and eventually also some bending stiffness.
UD laminate	Laminate with all fibres in the same direction, '**U**ni **D**irectional'.
pre-bend	Flapwise (for zero pitch) geometrical curvature of a blade without loading. Sometimes an up-wind pre-bend is used which partly eliminates the elastic deformation under normal operating loading. In Phatas the pre-bend is defined positive as the down-wind geometric shape of the blade tip.
Bending-Torsion coupling	Modification of the blade structure such that a bending moment gives a torsional deformation. This can be realised e.g. with off-axis UD laminates (symmetric w.r.t. the blade axis) or with an aft-swept blade tip.

1 INTRODUCTION

The phenomena of buckling is failure of a load-carrying structure by strong deformation that distorts the geometrical topology and the associated load-carrying capabilities. For analytical descriptions of buckling, one may start with the fundamental fact that the energy used for deformation of the geometry (elastic or plastic) is provided by the release of strain energy (usually compression) in the loaded structure.

The phenomena of buckling has been investigated for decades in aerospace industry; for the design of e.g. launch rockets and the upper skin of aircraft wings. These investigations were mainly addressed to aluminium (isotropic material) structures, eventually stiffened with stringers.

GARTEUR project

For buckling of fibre-reinforced composite structures fundamental investigations have been performed within the European GARTEUR group, that have been reported by B. Geier (DFVLR, Stuttgart) [8]. The investigations reported by Geier dealt with curved orthotropic panels loaded by compression. This comes close to the problem of buckling of rotor blades loaded by bending.

BUCKBLADE project

Partly based on the fundamentals of the work reported by Geier, the European research project BUCKBLADE was carried out, addressed to development and evaluation of different tools for the prediction of buckling of rotor blades. These methods included design rules (based on graphs and formulas from handbooks), tools for buckling of panels, tools for buckling of cross-sections of prismatic structures (such as rotor blades), and finite element packages. Part of this research project was a set of tests on some rotor blade structures at the University of Stuttgart. These rotor blade structures were designed by LM Glasfiber and built by LM AeroConstruct in the moulds of the former DEBRA blade.

STARION project

Partly as follow-up of the European BUCKBLADE project the Dutch STARION project was addressed to implement the tools from the BUCKBLADE project to be used for buckling load analyses of real rotor blade structures. For this purpose 3 tests were carried out on the outer parts of rotor blades. Also the panel-based method StaBlad (developed by ECN) and the tool for cross-sections Finstrip (formerly developed by SPE) were implemented in the design package Focus.

Scope of the BLADKNIK Investigations

The tools developed in the BLADKNIK project are implemented in the wind turbine design package Focus. The emphasis of the development was laid on the aspects that previously showed to be of importance for rotor blades. These aspects are 'including the longitudinal curvature', the detailed geometric (transverse) curvature and material distribution, and the buckling of sandwich panels. The buckling of sandwich panels was specifically addressed to the shear webs in a rotor blade.

A description of the engineering 'Design rules' is given in chapter 2, while the improved program Finstrip is described in brief in chapter 3. The description of a numerical prediction method for buckling of rotor blade panels are given in Appendix A, together with some theoretical background. The analytical solutions for panels with uniform curvature and material properties are

are described in Appendix B. The solutions in Appendix B appear to be compliant with several solutions from literature.

Both the Design rules described in chapter 2 (Farob method 1) and the analytical solution described in Appendix B assume uniform material properties for the entire panel. Panels within a rotor blade structure usually have a varying material distribution in transverse (width) direction. For these non-uniform distributions of geometry, material, and loading the panel average properties are calculated by integration using 'weighting functions' that are based on the deformation energy of the collapse mode. The weighting functions of this approach are described in Appendix C.

The improvements of the prediction tools with respect to the former BUCKBLADE project are summarised in chapter 4.

Some concluding remarks and items that can be improved are summarised in chapter 5.

For the buckling-load prediction of rotor blade panels, some FORTRAN routines were developed for use in Farob (Farob is the blade model under Focus) and for use in Crostab (a structural analysis tool for blade cross-sections). A description of the argument list and of the functionality of these routines is given in Appendix D.

2 'DESIGN RULES' FOR ORTHOTROPIC SANDWICH PANELS

This report describes the various types of buckling load prediction methods that are implemented in Farob or used within the wind turbine design package Focus. One of these methods are the here-called "Design rules" which are relatively fast and therefore suitable for scoping analyses to find the most unfavourable load combination or for analysing various stackings within the rotor blade panels.

The design rules developed in the BLADKNIK project are obtained by merging the most appropriate formulas from literature and by fitting them to calculated results from more sophisticated codes. Finally they apply to panels with:

Infinite length for which the buckling mode is the minimum for any half-wave length of the deformation pattern. The longitudinal edges are simply supported;

Orthotropic material properties This includes asymmetric laminates, viz for which the *B* matrix may be non-zero;

Sandwich panels with facings of similar material that may differ in thickness. This means that the membrane stiffness matrices of the facings are proportional to each other;

Transverse curvature only because longitudinal curvature increases the complexity of engineering solutions while not many publications deal with longitudinal curvature;

Combined loading in terms of linear varying longitudinal compression and in-plane shear load.

In addition to the Design rules for panels with simply-supported edges this chapter also contains the buckling load factors of panels with clamped edges. For clamped edge constraints no load interaction rules are given.

2.1 Load interaction rules

In the BUCKBLADE research project [13] the basis of the Design rules were the load interaction rules formulated by Bruhn [4], Plantema [20], and by Shanley [23].
These load interaction rules are in terms of the relative loading parameters for axial compression, transverse compression, in-plane bending, and shear:
$R_{\mathrm{c}} = N_{\mathrm{x}}/N_{\mathrm{x.cr}}$, $R_{\mathrm{t}} = N_{\mathrm{y}}/N_{\mathrm{y.cr}}$, $R_{\mathrm{b}} = N_{\mathrm{x\,bend}}/N_{\mathrm{x\,bend.cr}}$, and $R_{\mathrm{s}} = N_{\mathrm{xy}}/N_{\mathrm{xy.cr}}$.
These load interaction rules apply to panels with finite length, in which case the collapse modes for each of the individual load components have the same half-wave dimensions. For very long panels each of the load components may have a collapse mode with a different half-wave length. This means that the interaction for the different load components is less than for a plate with finite dimensions. For this reason predictions with these type of interaction rules for panels with combined loading may result in a (conservative) under estimation of the critical load!

The loading in panels of rotor blade cross sections is characterised by a varying axial loading combined with some amount of shear load. The latter is due to torsion and/or shear forces in the rotor blade. With a strong bending moment, the shear web of rotor blade sections may also be loaded by a transverse compressive loading: 'crushing loads'. This however was not yet modelled in version 5.1 of Focus5, it was found to be too complicated to include transverse compression properly in the load-interaction rules at that time.

2.1.1 Load interaction rules of Bruhn et al

For flat plates Bruhn, [4] ch. C.5, gives: $R_\mathrm{c} + R_\mathrm{b}^2 + R_\mathrm{s}^2 = 1$

For longitudinal compression and bending only Bruhn [4] gives $R_\mathrm{c} + R_\mathrm{b}^{1.75} = 1$ while Plantema [20] p.96 gives for sandwich panels $R_\mathrm{c} + R_\mathrm{b}^{1.5} = 1$. The latter relation is more conservative.

The here mentioned diversity in the contribution of in-plane bending of a panel was the reason to use the more detailed load interaction rule formulated in NEN6771 [19] for the combination of axial compression and in-plane bending, which is the characteristic loading in the contour panels of a rotor blade structure.

2.1.2 Load interaction rules of NEN6771

For flat panels with linear varying axial loading combined with uniform shear loading NEN6771 gives load interaction rules ((13.6-7) and (13.6-8) of [19]) that are compliant with the DASt Richtlinie 12). Although the load interaction rules in NEN6771 are presented for flat steel plates, they will be used as basis for the 'Design rules' presented here.

For a plate with linear varying longitudinal compression NEN6771 gives the critical value of the largest compressive loading in one of the panel edges.

The critical load factor itself is formulated in terms of the ratio ψ which is the smallest compressive load divided by the largest compressive load in the panel edges. For a compressive load distribution with $0 \le \psi \le 1$ (no tension in the edges) the critical load factor is

$$k_\mathrm{c} = 8.2/(1.05 + \psi)$$

and for a linear panel loading with tension in one of the edges $\psi < 0$ the critical load factor is

$$k_\mathrm{c} = 7.81 - 6.29\,\psi + 9.78\,\psi^2 \tag{1}$$

For the states of uniform compression (ψ = 1), for linear loading that is zero in one of the edges (ψ = 0), and of pure in-plane bending (ψ = -1) the critical load factors appear to be identical with the design rules given in 'Handbuch Struktur Berechnung' (H.S.B.) for orthotropic plates, [7].

2.1.3 Load interaction rules for strong curved panels

For curved panels in general the load interaction rules are hard to find, although literature gives load interaction rules for cylinders: e.g. Bruhn [4] ch.C.8. $R_\mathrm{c} + R_\mathrm{st}^2 + (R_\mathrm{b}^3 + R_\mathrm{s}^3)^{1/3} = 1$.
Here R_st is the torsional shear and R_b and R_s are bending and shear loading on the entire cylinder.

These and the previously presented rules show that different solutions apply for flat (or weak curved) panels and for panels with a strong curvature. This brings about the difficult choice whether the rules for flat panels or those for cylinders apply to curved panels. Initially this choice was made on basis of the dimensionless parameter Z. Finally this branch between weak curved and strong curved panels was eliminated by applying the load interaction rules for a so-called 'critical width' b_crit. This width is defined as the half-wave width of the collapse mode, of which it is trivial that it may not be larger than the panel-width.

For buckling of sandwich panels with a relatively soft core however, a branch remains for the face wrinkling mode.

2.2 Effective half-wave width of strong curved panels

In this subsection some solutions for buckling of curved panels are presented. For application of the 'Design rules' to buckling of curved panels, these solutions are used to express the curvature parameter Z to distinguish between weak and strong curved panels.

2.2.1 Buckling of curved isotropic sandwich panels

For long axially compressed curved isotropic sandwich panels Plantema, [20] p.179, gives:

$$\begin{aligned}
&\text{for} \quad Z/\pi^2 \le 4\,\frac{\sqrt{1-r}}{(1+r)^2} && k_c = \frac{4}{(1+r)^2} + (Z/\pi^2)^2\,(1-r)/4\,, \\
&\text{for} \quad 4\tfrac{\sqrt{1-r}}{(1+r)^2} \le Z/\pi^2 \le \tfrac{\sqrt{1-r}}{r} && k_c = \tfrac{2\,(Z/\pi^2)}{\sqrt{1-r}}\left(1-\tfrac{r}{4}\,\tfrac{2\,(Z/\pi^2)}{\sqrt{1-r}}\right)\,, \\
&\text{and for} \quad Z/\pi^2 \ge \frac{\sqrt{1-r}}{r} && k_c = 1/r\ .
\end{aligned} \tag{2}$$

The last expression of k_c is for the "shear buckling" mode.
Here Z is the curvature parameter $Z = \sqrt{1-\nu^2}\cdot b^2/(R_y\,h/2)$
and *r* is the stiffness ratio of sandwich panels $r = (\pi/b)^2\,D/(G_{23}\,h^2/t_c)$.

2.2.2 Buckling of curved orthotropic cylinders

For orthotropic cylindrical shells the curvature parameter Z has the form $Z = b^2/(R_y^*\sqrt{\tilde{D}_{22}\,C_{22}})$ with $1/R_y^*$ the 'effective curvature' that includes the asymmetry matrix K see (52) section B.1.

For cylindrical shell panels that are symmetric in thickness direction (a zero B matrix) Van der Neut [6] has published an approximate solution in which the critical load is related to the collapse load for axisymmetric buckling $N_{x.cr} = -\eta\,(2/R_y)\sqrt{\tilde{D}_{11}/C_{22}}$.
After adding the terms $\tilde{D}_{12}$, $\tilde{D}_{66}$, C_{12}, and C_{66} to this expression of V.d. Neut, the factor η can be calculated as

$$\eta = \frac{\sqrt{Y + 2\beta\sqrt{\frac{\tilde{D}_{22}\,C_{22}}{\tilde{D}_{11}\,C_{11}}} + \left(\frac{\tilde{D}_{22}\,C_{22}}{\tilde{D}_{11}\,C_{11}}\right)/Y}}{\sqrt{Y + \frac{(2\,C_{12}+C_{66})}{\sqrt{C_{11}\,C_{22}}} + 1/Y}}\ . \tag{3}$$

Following Van der Neut the minimum η is found near $Y = \sqrt{(\tilde{D}_{22}\,C_{22})/(\tilde{D}_{11}\,C_{11})}$.
If this minimum η is larger than 1 the shell fails in a mode that is (nearly) axisymmetric with a short longitudinal half-wave length at a load that corresponds with $\eta = 1$.
The buckling load factor k_c follows from $k_c = 2\,\eta\,(b/\pi)^2/(R_y\sqrt{\tilde{D}_{22}\,C_{22}}) = 2\,\eta\,Z/\pi^2$.
Combining this expression following Van der Neut with formula (11) for orthotropic sandwich plates to an expression of Plantema for long curved sandwich panels (formula (2)) gives:

$$\begin{aligned}
&\text{for} \quad \eta\,Z/\pi^2 \le k_{c\,flat}\sqrt{1-r} && k_c = k_{c\,flat} + \frac{(1-r)}{(2+2\beta)}\,\frac{\eta^2\,Z^2}{\pi^4}\,, \\
&\text{for} \quad k_{c\,flat}\sqrt{1-r} \le \eta\,Z/\pi^2 \le \frac{\sqrt{1-r}}{r_1} && k_c = 2\,\frac{\eta\,Z/\pi^2}{\sqrt{1-r}}\left(1-\frac{r_1}{2}\,\frac{\eta\,Z/\pi^2}{\sqrt{1-r}}\right)\,, \\
&\text{and for} \quad \eta\,Z/\pi^2 \ge \sqrt{(1-r)}\,/r_1 && k_c = 1/r_1\ .
\end{aligned} \tag{4}$$

The last expression of (4) describes the "shear buckling" mode, which is a failure mode for sandwich panels of which the facings have negligible bending stiffness. The half-wave length of the "shear buckling" mode is very small. If the facings have some bending stiffness, the sandwich panels may collapse in the so-called "face wrinkling" buckling mode, of which a description is given in section B.6. In the expressions (4) $k_{\mathrm{c\,flat}}$ is the buckling factor (11) for flat plates.

The expressions (4) are

- nearly exact for flat orthotropic sandwich plates;
- a close but conservative approximation for curved orthotropic panels;
- a good and conservative approximation for curved isotropic sandwich panels of which the core has moderate flexibility: $r << 1$;
- a reasonable (not always conservative) approximation for curved orthotropic sandwich panels.

2.2.3 Half-wave width considered

The load interaction rules in NEN 6771 [19] are formulated for flat panels, which implies that the collapse mode for axial compression has one half-wave in transverse direction. To have a physical basis to apply the load-interaction rules for linear varying axial loading, the 'Design rules' formulated here for curved panels consider a panel width that is (probably) the transverse half-wave dimension. For this purpose the critical half-wave width is chosen such that the first of the three expressions of (4) apply. This critical half-wave width is calculated from $\eta\, Z_{\mathrm{crit}}/\pi^2 \;=\; k_{\mathrm{c\,flat}}\,\sqrt{(1-r)}$. In this expression the averaged sandwich stiffness ratio r is calculated with the entire panel width (for simplicity). This condition for Z_{crit} gives an expression for the critical half-wave width b_{crit}

$$b_{\mathrm{crit}}{}^2 = Z_{\mathrm{crit}}\,|R^*_{\mathrm{y}}|\,\sqrt{\tilde{D}_{22}\,C_{22}} = \pi^2\,k_{\mathrm{c\,flat}}\,\sqrt{(1-r)}\,|R^*_{\mathrm{y}}|\,\sqrt{\tilde{D}_{22}\,C_{22}}\,/\eta\;. \tag{5}$$

Here $k_{\mathrm{c\,flat}}$ is evaluated with expression (11) for a flat sandwich plate.
If this critical half-wave width is larger than the panel width then the panel width is used.

If the critical half-wave width is smaller than the panel width the (linear varying) loading is evaluated on the panel edge with the largest compression. If for a linear varying loading the load halfway b_{crit} is tension, the panel width b_{crit} is reduced further until the loading halfway b_{crit} is zero.

2.3 Buckling of axially compressed panels

2.3.1 Axially compressed orthotropic plates

Linear varying longitudinal compression can be decomposed in uniform axial compression and in-plane bending. The load-interaction rules in NEN6771 however give a critical load factor for an arbitrary linear varying axial compression, which is related to the largest compressive stress in one of the panel edges. This factor is expressed for steel plates, which are isotropic. In this subsection the buckling factors of orthotropic plates found in literature are described for some characteristic load combinations such as uniform compression, linear compression that is zero in one of the edges, and pure in-plane bending. The resulting buckling factors will then be merged with the expressions given in NEN6771 for the critical load factor.

Uniform longitudinal compression (ψ = 1)

For long orthotropic plates with simply supported edges subjected to uniform axial compressive loading the critical axial compressive load can be derived easily from the stability equations (see (59) in section B.4 for r_1 = r_2 = 0) which gives: $N_{\mathrm{x.cr}} = -k_{\mathrm{c\,flat}}\,(\pi/b)^2\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}$ where the factor $k_{\mathrm{c\,flat}}$ is

$$k_{\mathrm{c\,flat}} = 4 + 2\,(\beta - 1)\ . \tag{6}$$

For long orthotropic plates with clamped edges the relation from 'Handbuch Struktur Berechnung' (H.S.B.) [7] (see also section 4.1.1 of the BUCKBLADE report [13]) holds.

$$k_{\mathrm{c\,flat}} = 6.97 + 2.36\,(\beta - 1)\ .$$

Linear varying axial compression that is zero in one edge (ψ = 0)

For long flat orthotropic panels with simply-supported edges loaded with a linear increasing load that is zero in one of the edges, H.S.B. [7] gives a graph with the critical load factor that can be fitted with:

$$k_{\mathrm{c\,flat}} = 7.81 + 3.85\,(\beta - 1)\ . \tag{7}$$

For isotropic plates ($\beta = 1$) this expression gives $k_{\mathrm{c\,flat}}$ = 7.81 which equals the factor in the load interaction rules of NEN6771 [19] for $\psi = 0$, see (1) in section 2.1.2.

For long isotropic plates with the same loading but with clamped edges [7] gives $k_{\mathrm{c\,flat}} = 13.54$.

In-plane bending of orthotropic plates (ψ = -1)

For long flat orthotropic panels loaded with in-plane (edgewise) bending the following rule can be derived from the graphs in H.S.B. [7] and in NASA TP3568 [18] for simply supported edges

$$k_{\mathrm{c\,flat}} = 23.88 + 10.5\,(\beta - 1) \tag{8}$$

and for clamped edges

$$k_{\mathrm{c\,flat}} = 39.52 + 12.8\,(\beta - 1) - 0.2\,(\beta - 1)^2\ .$$

Arbitrary linear compressive loading

Following the formulation given in NEN6771 the factor for linear varying axial compression is defined as function of the parameter ψ, which is valid for ψ larger than -1. For panels of which the loading is described with ψ smaller than -1, the panel width is reduced (on the tension side) such that the remaining load distribution applies to $\psi = 1$. The expressions (13.6-7) and (13.6-8) in

NEN6771 [19] apply to load distributions for $\psi \leq 0$ and for $\psi > 0$ respectively. For orthotropic plates the contribution of the orthotropy (represented by $(\beta - 1)$) is included in the expressions from NEN6771 on a similar way as the factor for orthotropic plates. For $\psi \geq 0$ the result is:

$$k_{\mathrm{c\,flat}} = 8.2/(1.05 + \psi) + (\beta - 1)\,4.16/(1.08 + \psi) \tag{9}$$

and for $\psi < 0$ the result is:

$$k_{\mathrm{c\,flat}} = 7.81 - 6.29\,\psi + 9.78\,\psi^2 + (\beta - 1)\,(3.85 - 3\,\psi + 3.65\,\psi^2)\;. \tag{10}$$

The coefficients 3 and 3.65 in the orthotropy term of (10) are estimated, but are correct for the case of pure in-plane bending (ψ = -1) and linear increasing axial compression (ψ = 0).

2.3.2 Axially compressed sandwich plates

The influence of the out-of-plane shear flexibility is expressed as a reduction factor on the buckling coefficient $k_{\mathrm{c\,flat}}$ which is a function of the sandwich stiffness ratios r_1 and r_2. The definition of these stiffness ratios fits to the analytical solution presented in section B.4, see (57).

Uniform axial compression (ψ = 1)

For axially compressed sandwich plates with small values of r_1 and r_2 the linearised analytical solution of the buckling factor was derived in section B.4 see (59) in terms of a 'reduction factor' on $k_{\mathrm{c\,flat}}$: $1/(1 + r_1 + r_2)$. For axially compressed sandwich plates Wiggenraad reported a 'reduction factor' of $(1 - r_1 + r_2)/(1 + 2\,r_2 + r_2^2)$. For small values of the stiffness ratios this expression fits the analytical solution of (59). When considering also quadratic terms in the stiffness ratios, the 'reduction factor' of Wiggenraad fits very close to the empirical relation that has been formulated in the former BUCKBLADE project [13] which gives:

$$k_{\mathrm{c\,flat}} = (\,4 + 2\,(\beta - 1)\,)/(\,1 + r_2 + r_1 + r_1^{\,2}\,)\;. \tag{11}$$

Compared to the solution of Wiggenraad, the expression (11) is numerically "more robust" for large values of r_1 because the factor $k_{\mathrm{c\,flat}}$ can not become negative.

For long orthotropic plates with clamped edges the relation from H.S.B. [7] (see also section 4.1.1 of [13]) holds. Expanding this relation such that it fits to Figure 5.8 on p.137 of Plantema [20] gives the expression for sandwich plates

$$k_{\mathrm{c\,flat}} = (6.97 + 2.36\,(\beta - 1)) \cdot \frac{(1 + r_1)}{(1 + 3.5\,r_1 + 3\,r_2 + 7\,r_1^{\,2})}\;.$$

Linear varying compression that is zero in one edge (ψ = 0)

For long orthotropic sandwich plates with simply supported edges subjected to uniform axial compressive loading the critical load was calculated with the 'rigorous solution method' described in A.4 and A.5 for different values of the sandwich out-of-plane shear flexibilities. From the relation between the calculated load factors and the dimensionless stiffness ratios r_1 and r_2 the following conservative expression for the buckling load factor was formulated:

$$k_{\mathrm{c\,flat}} = (7.81 + 3.85\,(\beta - 1)) \cdot \frac{(1 + 0.3\,r_2)}{(1 + r_1 + 1.4\,r_2 + 1.1\,r_1^{\,2} + 1.25\,r_1\,r_2)}\;. \tag{12}$$

In plane bending (ψ = -1)

For long sandwich plates with an orthotropic core and simply-supported edges loaded by in-plane bending Plantema, [20] p.145 Fig.5.14, gives a graph for the buckling-factor that can be approximated conservatively with $k_{\mathrm{c\,flat}} = k_{\mathrm{b}} = 23.88 \cdot (1 + 6\,r_1)/(1 + 6.5\,r_1 + 4\,r_2 + 80\,r_1^{\,2})$.

For isotropic sandwich panels with stiffness ratio $0.1 < r < 0.25$ this relation gives an under-estimation of up to 12%. For other values of the stiffness ratio r this function gives a close approximation. For a (flexible) core with a large value of the stiffness ratio r_1 the panel fails in a "shear buckling" mode for which Plantema gives $k_{\mathrm{b}} = 1.886/r_1$.
Combining the expression from H.S.B. from [7] with the empirical fit for the influence of the sandwich flexibility properties gives:

$$k_{\mathrm{c\,flat}} = k_{\mathrm{b}} = (23.88 + 10.5\,(\beta - 1)) \cdot \frac{(1 + 6\,r_1)}{(1 + 6.5\,r_1 + 4\,r_2 + 80\,r_1^{\,2})} \; . \tag{13}$$

Note that for large values of r_1 expression (13) approaches $k_{\mathrm{c\,flat}} = 1.791/r_1$.

Arbitrary linear compressive loading

For a linear varying axial loading that is compressive over the entire panel width (so $\psi > 0$) the factor for the sandwich shear flexibilities is interpolated linearly between the expressions for $\psi = 0$ and $\psi = 1$, which gives for the buckling factor:

$$k_{\mathrm{c\,flat}} = k_{\mathrm{c\,(non-sandwich)}} \left(\frac{\psi}{1 + r_2 + r_1 + r_1^2} + \frac{(1-\psi)\,(1 + 0.3\,r_2)}{1 + r_1 + 1.4\,r_2 + 1.1\,r_1^2 + 1.25\,r_1\,r_2} \right) \tag{14}$$

For axial compressive loading that is tension in one of the edges ($\psi \leq 0$) interpolation of the influence of the sandwich flexibilities between expressions for $\psi = 0$ and $\psi = -1$ gives:

$$k_{\mathrm{c\,flat}} = k_{\mathrm{c\,(non-sandwich)}} \left(\frac{(1+\psi)\,(1 + 0.3\,r_2)}{1 + r_1 + 1.4\,r_2 + 1.1\,r_1^2 + 1.25\,r_1\,r_2} - \frac{\psi\,(1 + 6\,r_1)}{1 + 6.5\,r_1 + 4\,r_2 + 80\,r_1^2} \right) \tag{15}$$

2.3.3 Curvature effects

The effects of curvature are included using the expressions of Van der Neut in terms of Z for slightly curved panels (4). This is done independent from the detailed axial load distribution:

$$k_{\mathrm{c}} = k_{\mathrm{c\,flat}} + \frac{(1-r)}{(2 + 2\,\beta)} \, \frac{\eta^2\, Z^2}{\pi^4} \; .$$

After evaluation of this expression for buckling of curved sandwich panels, one should check whether the panel fails in a face wrinkling mode with a short half-wave length such as described in section B.6. Based on the fact that this mode has a short half-wave length the critical load factor for face wrinkling has to be evaluated with the largest compressive load in one of the edges.

2.4 Buckling of shear loaded sandwich panels

For long orthotropic plates loaded by shear the graphs in NASA TP3568 [18] can be fitted with
$k_\mathrm{s} = 5.35 + 1.7\,(\beta - 1) - 0.1\,(\beta - 1)^2$ for simply supported edges, and with
$k_\mathrm{s} = 8.99 + 2.6\,(\beta - 1) - 0.17\,(\beta - 1)^2$ for clamped edges.
For long flat orthotropic sandwich plates, Figure 5.12 on p.142 of Plantema [20] can be fitted with
$k_\mathrm{s} = 5.35\,(1 - 0.28\,r_1 + 0.1\,r_2)/(1 + 2.9\,r_2 + 0.5\,r_1)$.
Combining these expressions gives

$$k_\mathrm{s} = [5.35 + 1.7\,(\beta - 1) - 0.1\,(\beta - 1)^2] \cdot \frac{(1 - 0.28\,r_1 + 0.1\,r_2)}{(1 + 2.9\,r_2 + 0.5\,r_1)} \ . \tag{16}$$

For shear loaded plates with clamped edges the empirical relation (14) on p.92 of Plantema [20] (for isotropic sandwich plates) can be combined with the graphs in TP3568 to

$$k_\mathrm{s} = [8.99 + 2.6\,(\beta - 1) - 0.17\,(\beta - 1)^2] \cdot \frac{(1 - 0.12\,r)}{(1 + 7\,r)} \ .$$

Curvature effects

For buckling of long isotropic cylindrical panels loaded by torsional shear Timoshenko & Gere [25] p.489 give the empirical formula based on many calculations
$\tau_\mathrm{cr} = 4.82\,(t/b)^2\,E\,\sqrt[4]{1 + 0.0145\,b^4/(R_\mathrm{y}\,t)^2}$.
On p.C.46 of Dubbel [2], a similar expression is given with the factor 0.0146 instead of 0.0145.
Re-writing the more conservative expression of Timoshenko & Gere in terms of the curvature parameter Z (for $\nu = 0.295$) and the bending stiffness D gives the expression for the shear-buckling factor: $k_\mathrm{s} = 5.35\,\sqrt[4]{1 + 0.0013\,Z^2}$.

Gerard & Becker [10] give design graphs that are also published by Bruhn, [4] C9.4 and C9.2. The graph C9.4 for simply supported (**SS3**) edges corresponds closely with the formula given by Timoshenko & Gere while the graph C9.2 for clamped (**C4**) edges can be approximated with
$k_\mathrm{s} = 8.98\sqrt[4]{1 + 0.0014\,Z^2}$.
This expression for clamped edges always gives an under-prediction not larger than 11% compared with the graph C9.2 of Bruhn [4].
(Note that in Bruhn the curvature parameter used is $Z_\mathrm{b} = (b^2/(R_\mathrm{y}t)) \cdot \sqrt{1 - \nu^2}$.)

Combining the expressions for flat orthotropic sandwich plates with those for curved isotropic panels with simply-supported edges gives

$$k_\mathrm{s} = [5.35 + 1.7\,(\beta{-}1) - 0.1\,(\beta{-}1)^2]\,\frac{(1{-}0.28\,r_1{+}0.1\,r_2)}{(1{+}0.5\,r_1{+}2.9\,r_2)} \cdot \sqrt[4]{1{+}0.0013\,Z^2} \tag{17}$$

and for clamped edges

$$k_\mathrm{s} = [8.99 + 2.6\,(\beta{-}1) - 0.17\,(\beta{-}1)^2]\,\frac{(1{-}0.12\,r)}{(1 + 7\,r)} \cdot \sqrt[4]{1{+}0.0013\,Z^2} \ .$$

These expressions are accurate for moderate values of the sandwich stiffness ratios: r_1 , $r_2 < 1$.

N.B.

For long cylindrical isotropic panels loaded by torsional shear Roark & Young [22] give the empirical rule $\tau_\mathrm{cr} = E \cdot (0.1\,t/R_\mathrm{y} + 5\,(t/b)^2)$ for simply supported edges
and $\tau_\mathrm{cr} = E \cdot (0.1\,t/R_\mathrm{y} + 7.5\,(t/b)^2)$ for clamped edges.
For values of Z between 30 and 500 the expression in Roark and Young is un-conservative compared with the expressions given in section 7.3.2 of Dubbel [2] and by Gerard & Becker [10].

2.5 Half-wave length and orientation of the collapse mode

The 'Design Rules' described in this chapter are implemented in the routines/functions *bucpan3* and *bucweb2* that are included in the Focus design package (linked to Farob). To return information to the Focus user-interface about the failure mode of the rotor blade structure, expressions are implemented for the half-wave length and the orientation of the deformation pattern.

Axial compression

In section B.4 an analytical solution is derived (58) for the critical half-wave length of sandwich plates loaded by uniform axial compression:

$L_{\mathrm{c\,crit}} = b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}\,(1 + r_2/2)/(1 + r_1 + \beta\, r_2/2)$.

This solution was linearised for small values of r_1 and r_2. For small values of the sandwich stiffness ratios, this solution fits to the expression of Plantema, on p.81 of [20]:

$L_{\mathrm{c\,crit}} = b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}\,\sqrt{(1-r)/(1+r)}$.

Note that this expression is invalid if the sandwich stiffness ratio r is larger than 1.

For plates with linear varying longitudinal compression that is zero in one of the edges ($\psi = 0$) the graphs in H.S.B. [7] show that the critical half-wave length is nearly the same as for uniform axial compression. For (non-sandwich) plates loaded by in-plane bending ($\psi = -1$) the graphs in H.S.B. [7] and NASA TP3568 [18] give $L_{\mathrm{c\,crit}} = 0.674\, b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}$.

For plates loaded by linear axial compression that is tension in one of the edges ($\psi < 0$) the critical half-wave length is interpolated linearly as function of ψ:

$$L_{\mathrm{c\,crit}} = (1 + 0.33\,\psi)\, b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}\,(1 + r_2/2)/(1 + r_1 + \beta\, r_2/2)\ . \tag{18}$$

Shear loading

For shear loaded (non-sandwich) plates Timoshenko & Gere ([25] p.383) present an approximate analytical solution that gives a critical half-wave length of $L_{\mathrm{s\,crit}} = \sqrt{1.5}\, b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}$.
For the 'Design Rules' described here the factor in this expression is approximated with 1.225.
The critical half-wave length of shear loaded (isotropic) sandwich plates is given in Figure 4.4 on p.92 of [20] which fits to the relation $L_{\mathrm{s\,crit}} = 1.225\, b\,(1 - 0.56\, r)/(1 + 0.62\, r)$.
This dependency on the sandwich stiffness ratio will be approximated with a factor $1/(1 + r_1)$ because for axial compression the dependency on r_2 shows to be weak. In addition the latter expression also gives a positive half-wave length for large values of r_1 .

Combining this expression with that for the half-wave length of orthotropic panels gives:

$$L_{\mathrm{s\,crit}} = 1.225\, b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}\ /(1 + r_1)\ . \tag{19}$$

Orientation of the collapse mode

For axial compressive loading, the direction of the (periodic) deformation pattern has a 'zero' angle with respect to the transverse direction $\theta_{\mathrm{c\,crit}} = 0$.

For shear loading the deformation pattern is skewed with respect to the transverse direction.
For shear loaded plates this skew angle is $\theta_{\mathrm{s\,crit}} = \arctan(\sqrt{0.5}) \approx 35.3^{\mathrm{o}}$.

Curvature effects

The influence of curvature on the critical half-wave length and the orientation of the collapse mode is not included in the 'Design rules' presented here because it was too complicated to derive.

2.6 Routine for contour panel buckling

The more simplified but faster routines for buckling analyses of rotor blade sections are based on buckling of the 'panels' between the shear webs or between the shear web and the trailing-edge of the airfoil. This simplification implies that the interactions of the panel-edges with the complete cross-sectional structure are omitted. If the panel edges are treated as simply-supported ('hinged') the resulting solution is an under-estimation of the buckling load because stiffness at the edges is omitted. The advantage of this simplification is a reduction in computing time for a reduction of accuracy that is conservative.

For use by Farob (under Focus) two routines with panel-based buckling load prediction methods were developed. One of these routines was developed for buckling of non-uniform and curved contour panels while another routine was developed for buckling of a flat shear web. The latter routine uses different load-interaction rules because the shear web loading is dominated by in-plane bending and in-plane shear, see section 2.7.

The simplest buckling prediction method implemented in the panel-based buckling routines is based on the 'Design rules' that are described in the former sections.

The routines that are based on the 'Design rules' use panel-average material properties and panel-average loading. The calculation of these properties is described in Appendix C. Compared to the shear web, the contour panels of a rotor blade can have a non-uniform curvature that is stronger near the leading edge. The loading of the contour panels is dominated by (non-uniform) axial compression. Realise that panels loaded by tension are not buckling critical.

The critical load factor is calculated as a factor λ on the applied axial and shear loading for which buckling occurs. The non-uniform axial compression is expressed in terms of a uniform loading and a (linear) in-plane bending, see section C.3. For the resulting linear load distribution the critical value is calculated based on the expressions in NEN6771. The relatively small shear loading in the contour panels is taken into account following the load-interaction rules from Bruhn ([4] expression C5.17 on p.C5.9) for cylindrical curved panels under axial compression and shear:

$$\lambda\, N_{\mathrm{x}}/N_{\mathrm{x.cr}} + \lambda^2\, (N_{\mathrm{xy}}/N_{\mathrm{xy.cr}})^{\,2} = 1\ . \tag{20}$$

From this expression the factor λ is solved, which is considered the 'critical load factor'.
The load-interaction rule (20) is conservative for slightly curved or flat panels.

Half-wave length for axial compression and shear

For combined axial compression and shear the critical half-wave length is calculated by interpolation between the values for compression $L_{\mathrm{c\,crit}}$ and shear $L_{\mathrm{s\,crit}}$. This interpolation is on basis of the value of the load components relative to their critical values:

$$L_{\mathrm{crit}} = (1 - frac)\, L_{\mathrm{c\,crit}} + frac\; L_{\mathrm{s\,crit}}\ .$$

In this expression, $frac$ is calculated from

$$frac = \frac{(N_{\mathrm{xy}}/N_{\mathrm{xy.cr}})^2}{(N_{\mathrm{xy}}/N_{\mathrm{xy.cr}})^2 + N_{\mathrm{x}}/N_{\mathrm{x.cr}}}\ .$$

2.7 Routine for shear web buckling

The shear web is characterised as a flat plate with usually a symmetric layup. The loading is dominated by in-plane bending and shear, while there may be a (non-linear) transverse compressive loading; 'crushing load'. The latter compressive loading is included in the routines developed in the 'BLADKNIK' research project.

Similar as for the contour panels, the critical value for the varying axial load distribution is calculated on basis of the load interaction rules from NEN6771. Using the fact that the loading in the shear web is dominated by in-plane bending and shear loading, the critical value for the combined loading is evaluated from the load interaction rule in Bruhn ([4] expression C5.16 on p.C5.8) :

$$\lambda^2 (\,(N_{\rm x}/N_{\rm x.cr})^{\,2} + (N_{\rm xy}/N_{\rm xy.cr})^{\,2}) = 1 \;. \tag{21}$$

In this relation $N_{\rm x}$ is the largest compressive loading in any of the panel edges.

Including transverse (compressive) loading

For buckling of an orthotropic flat plate with simply supported edges under in-plane bending and transverse compression Michael Nementh gives a graph in Figure 7 of [18] that can be fitted with the empirical relation: $(N_{\rm x}/N_{\rm x.cr}) = 1 + 9.46\,\left(\sqrt{1-(N_{\rm y}/N_{\rm y.cr})/2}\,-1\right)/k_{\rm c\,flat}$.

Here $k_{\rm c\,flat}$ is according to (8) and $N_{\rm x}$ is the largest compressive load in one of the edges. Assuming that the in-plane bending moment has roughly the same interaction with transverse compressive loading, the contribution of the in-plane shear loading can be included as follows:

$$\lambda\sqrt{(N_{\rm x}/N_{\rm x.cr})^{\,2} + (N_{\rm xy}/N_{\rm xy.cr})^{\,2}} = 1 + 9.46\,\left(\sqrt{1-\lambda\,(N_{\rm y}/N_{\rm y.cr})/2}\,-1\right)/k_{\rm c\,flat} \;. \tag{22}$$

Note that in this expression the relative loading terms are multiplied with the load factor λ.
A solution for λ is obtained numerically from this equation. If the critical value of λ appears to be larger than $N_{\rm y.cr}/N_{\rm y}$ and the transverse loading is compressive then the failure mode is dominated by the transverse compressive loading and thus prismatic. For this case the critical value for λ is $N_{\rm y.cr}/N_{\rm y}$ while the half-wave length of the collapse mode is set to the length of the panel, and the orientation of the collapse mode is set to $\pi/2$ rad = 90deg.

Orientation of the half-waves for axial compression and shear

For the combined axial compressive (also in-plane bending) and shear loading in the shear web, the orientation of the deformation pattern is calculated from the relative fractions of these load components. This is done with a similar 'interpolation' as for the half-wave length:

$$\theta_{\rm crit} = (1-frac)\,\theta_{\rm c\,crit} + frac\;\theta_{\rm s\,crit} \;.$$

Similar as for the half-wave length, $frac$ is calculated from

$$frac = \frac{(N_{\rm xy}/N_{\rm xy.cr})^2}{(N_{\rm xy}/N_{\rm xy.cr})^2 + (N_{\rm x}/N_{\rm x.cr})^2} \;.$$

3 PROGRAM FOR PRISMATIC STRUCTURES FINSTRIP

The program Finstrip is developed at STORK Product Engineering, Amsterdam. With the program Finstrip buckling loads of prismatic beams of orthotropic material can be predicted. In Finstrip the cross section of the beam (e.g. a rotor blade) is described with a finite number of strips that can have the material properties of sandwich- and/or single laminates. The cross sectional loading is longitudinal compression resulting from a longitudinal force and sectional bending moments.

Meshing of the model is performed automatically by Finstrip for a user-defined maximum element length. The meshed section can be checked visually. For every load case and half-wave length bifurcation factors are calculated as the buckling load divided by the applied load.

The bifurcation factors are displayed graphically as a function of the half-wave length. For a selected half-wave length graphs are shown of the undeformed- and deformed section. To ease evaluation of the results and to check the input data, some information can be displayed and printed or saved to text files. Among others this information contains the element mesh, the section inertia and the strains. The output of the program Finstrip includes a graph of the deformed cross section showing both the modelling of the cross section and the collapse mode.

The solution method of the program Finstrip is based on the following assumptions:

- Prismatic structure without geometric imperfections;
- Only initial buckling is predicted;
- Non-linear material properties (such as local composite failure and plastic material behaviour) are not taken into account;
- Only symmetric orthotropic laminate properties. In the A, B, and D matrices, only elements A_{11}, A_{12}, A_{22}, A_{66}, D_{11}, D_{12}, D_{22}, and D_{66} are used;
- The material is modelled symmetric with respect to the defined geometry;
- The load distribution along the blade axis is according to engineering bending theory for thin-walled sections, such that only normal strains and -stresses are considered (no shear);
- The initial curvature due to bending of the beam is not accounted for.

With the program Finstrip buckling of the 2D model of the cross section can be solved including the complete interaction of all panels in the cross section. Compared to a non-linear finite element code the program requires less input properties and is suitable for the (pre-) design of rotor blades.

Finstrip is distributed as part of the Focus Integral Wind Turbine Design Tool. After Stork stopped all activities on Wind turbine projects, Knowledge Centre WMC started maintaining the software. In 2002 an update of Finstrip was released by WMC with the following modifications:

- The load axis system was modified into the system recommended by Germanischer Lloyd.
- Optional, blade stiffnesses as calculated by Focus can be used within Finstrip analyses. Since Finstrip puts the nodes of the elements in the centre of the layup, half of the thickness is out-side the blade contour, thus calculating a too high stiffness.
- All calculation output is now automatically saved to disk after an analysis is completed. Originally, all output files had to be saved manually.

Detailed information on Finstrip is provided in the 'Theoretical Reference' [27] and in the 'Users Manual' [26] which are both part of the Focus documentation.

4 IMPROVEMENTS OF THE TOOLS

4.1 Improvements on the panel-based prediction methods

Improvements on the Design rules

The research within the BLADKNIK project was addressed to evaluation and further development of tools for the buckling load prediction of complete rotor blades. This implies that only programs and finite-element codes for buckling load prediction of complete cross-sections were investigated. In (classical) practise many engineers use formulas from design handbooks for buckling load predictions of flat panels and strips, usually for a quick evaluation of the buckling strength in the pre-design. For this reason and also because a lot of knowledge on design rules has already been gained in the (former) BUCKBLADE project, these 'Design rules' were still involved in the BLADKNIK project.

The improvements of the 'Design rules' are:

- For linear axial loading, the expressions from NEN6771 are used instead of the more rough interaction rules for axial compression and in-plane bending.
- The definition of the sandwich shear stiffness ratios is revised on basis of analytical solutions for long orthotropic plates, see section B.4.
- The critical load factor for orthotropic sandwich plates is improved, see section B.4.
- Strong curved panels are replaced by smaller panels on basis of a critical half-wave width. This was done because most Design rules apply to flat (or weak curved) panels.
- The contribution of the terms K_{ij} of the asymmetry matrix K are added to the so-called 'effective curvature': $1/R^*_{\mathrm{y}}$, see (52) in section B.1.
- Expressions are added for the critical half-wave length.

In addition to these improvements, the panel average loading, curvature, and material properties are used instead of properties for the centre of the panel, see Appendix C. Although relatively simple, the use of panel average properties is considered a practical and useful improvement compared to the use of stiffness properties in the centre of the panel (as in the Stablad routines developed in the former BUCKBLADE project).

Improvements on the analytical panel-based solution

The analytical panel-based solutions have not improved substantially, except for the fact that they are made more robust in the sense that the less reliable influence of anisotropic material properties has been simplified to the contribution following Tennyson e.a [24], see also section B.2. Other improvements are:

- Use of panel-average material properties, see Appendix C;
- Use of NEN6771 to account for non-uniform axial loading;
- Use a single parameter for the transverse curvature, evaluated with a more conservative expression, see section C.4.

In addition to the routine with an analytical solution, a routine was developed that includes the non-uniform geometry and material properties. At the moment of completion of the BLADKNIK project this routine worked well for flat panels.

4.2 Improvements on Crostab

Status of release "SEP-2002"

The program Crostab was developed at ECN Wind Energy for the prediction of buckling of thin-walled multi-cell beams. Similar as Finstrip, Crostab can be used for buckling load calculations of thin-walled prismatic beams, including the geometric longitudinal curvature from bending and the effects of crushing loads. Crostab solves the buckling of a cross section by stability analyses of each of the individual panels, where Finstrip models the complete structural integrity within a cross-section. As for Finstrip, small modifications have been issued to the program since the STARION project. The calculations for the State of the Art report [16] were performed with Crostab release "SEP-2002" of which the definition of materials was made more robust in the winter of 2005.

Improvements of release "SEP-2005"

Because the program Crostab does not play a vital role in the Focus design tool, its development as buckling load prediction tool was not given priority. Its function as tool for the calculation of the sectional properties of multi-cell beams of orthotropic material was still maintained because the layup of the panels was treated following laminate theory with which it is a good reference for the calculation of sectional properties.

In release "SEP-2005" of Crostab the 'new' routine '*panini2*' (see section D.1) is linked. In the call by Crostab the in-plane flexibility matrix $C = A^{-1}$ is returned, which is used to calculate the sectional stiffness properties.

4.3 Improvements on Finstrip

The program Finstrip was originally developed at Stork Product Engineering by M.J. van Varik, and was developed further by WMC, see chapter 3. The program Finstrip can be used to calculate the buckling load of multi-cell thin-walled prismatic beams. The analytical model does not account for geometric non-linear effects such as the longitudinal curvature under loading and the deformation from 'crushing loads'. After the former projects on buckling load prediction methods 'BUCKBLADE' and 'STARION' some details of Finstrip have been changed. The calculations for the State of the Art report [16] were performed with version 1.5 of Finstrip.

Faster eigenvalue solver

In the initial Finstrip versions an eigenvalue solver was used (based upon the Choleski eigenvalue solver from "Numerical Recipes" [21]), which resulted in long calculation times (could be several minutes or longer). Within the BLADKNIK project a more dedicated solver (based upon DSTEIN of the Lapack numerical library of `www.netlib.org`) was implemented that searches for the smallest eigenvalue only. This solver is based on the fact that the stiffness matrix is symmetric and positive definite and the initial stiffness matrix is symmetric not positive-definite. This new solver resulted in an up to 20 times faster performance of Finstrip, which allows analyses of more detailed cross-sections and a more interactive design process.

Use the reduced stiffness matrix

The program Finstrip models the material symmetric with respect to the geometric contour. For the description of a rotor blade such as in Farob (and in many wind turbine design applications) the material is defined 'inside' of the blade contour which means that with Finstrip the overall bending stiffness of the blade cross section would be over-estimated. This over-estimation of the bending stiffness has been by-passed by writing more realistic sectional properties from Farob to the `.buc` files.

Even with more realistic sectional properties, the B matrix that exists for asymmetric rotor blade panels is not included. Within the BLADKNIK project the asymmetry of rotor blade panels is included by using the 'reduced stiffness matrix' $\tilde{D}$ which is calculated as $\tilde{D} = D - B \cdot A^{-1} \cdot B$, see also section A.6.

Improvement on sandwich panel modelling

At the end of the former BUCKBLADE project [13], some uncertainty remained about the modelling of sandwich panels. Initially this uncertainty was addressed to the transitions between sandwich 'elements' and conventional 'elements'.

Scoping investigations for a single box-beam with various sandwich layup showed unrealistic results for sandwich panels if the sandwich core has an in-plane stiffness in the order of the facing stiffnesses such that the core contributes to the panel bending stiffness.

Correction of the sandwich modelling may be done with the expressions for the out-of-plane shear flexibility of the panel as calculated following the method in Appendix A.2, and by calculating the bending stiffnesses as for a conventional panel.

4.4 Improvements on Farob (Focus)

Modelling of the transverse shear force in a section

In former versions of Farob only the axial strains (and the associated stresses) in a blade cross section were calculated from the axial force and the bending moments. Within the BLADKNIK project the model for cross-sectional loads in Farob was extended with the calculation of the shear strain distribution in the panels of a blade cross section.

Option to define 'Lines' in a flat plane through a blade

In the Farob blade model material layers and shear webs are defined along so-called 'Lines'. These Lines are defined along a number of points on the contour of a series of cross sections. For the edges of a shear web ('spar') these Lines are the intersection of the overall blade contour and of the plane of the shear web, which is usually flat. For the intersection of the overall blade contour with a flat plane the Lines can be defined with the `/P` option that has been added to Farob.

Specification of material type

Within Farob the definition of materials is rigorously restructured such that the material type can be specified explicitly. At the DEF MATERIAL input specification the (optional) keyword *Type* has been added which can have the values:

'isotropic' Here *E_11* and *nu_12* are read and *G_12* is calculated by Farob;

'orthotropic' Here *E_11*, *E_22*, *G_12*, *nu_12*, and the fiber-angle are read;

'core' Here *E_11*, *E_22*, *G_12*, *G_23*, *G_31*, *nu_12*, and the fiber-angle are read.

These material properties can be extended to fully anisotropic (also *E_33*, *nu_23*, and *nu_31*).

If *Type* is specified, a check is performed whether all required material properties are present.
If *Type* is not specified the material properties are read in the conventional way (to ensure compatibility with previous versions).

Although not all of these properties are used by Farob they are used in the input files generated for FEM calculations. MSC.MARC, MSC.Nastran, and ANSYS all use these material properties.

For conventional strength analyses in Farob (fatigue/stress/strain analyses) the material properties regarding stiffness are transformed to the blade-longitudinal directions for non-zero fiber angle.

4.5 Improvements on the Focus-FEM interface

Farob-FEM interface for better non-linear elements

On request of Focus users, the finite element mesh generator was improved to have an additional option for ANSYS output to mesh 'SHELL181' elements, which is a composite 4-node thick shell element. For non-linear analyses in rotor blade certification (such as buckling) the usage of the 8-node thick shell element SHELL99 of ANSYS is no longer accepted by Germanischer Lloyd because of inaccuracies of SHELL99 for such analyses.

5 CONCLUDING REMARKS

5.1 Specification of the tools

The following table contains the specifications of all the buckling load prediction tools, that were improved in the BLADKNIK project and are supported by Farob. The numbers preceding the panel-based methods indicate the 'method' as used in Farob, see also section D.2.

	1.contour	1.web	2.analytic.	3.rigorous	Finstrip	F.E.M.
Asymm. orthotr. material	•	•	•	•	approx.	•
Anisotropic material			C_{16} C_{26}			•
Thick-faced sandwich	•	•	•	•		•
Asymmetric sandwich				approx.		•
Non-uniform material	aver.	aver.	aver.	•	•	•
Shear loading	•	•				•
Transverse compression		•	•			•
Non-uniform transv. curv.				(•)	•	•
Strong transverse curv.				(•)	•	•
Longitudinal blade curv.			•	•		•
Pre-buckling deformation						•
Geometric twist						•
Edge constraints	SS-3	SS-3	SS-3	SS-3	full	full

This table emphasizes the completeness of using a F.E. method for buckling load prediction.

When finishing the BLADKNIK project the 'rigorous solution' for panels with non-uniform geometry, loading, and material properties was only completed for flat panels; see the dots between brackets (•). In the BLADKNIK project description however, it was stated that the non-uniform properties of rotor blade panels would be included by either using panel-average properties or developing the 'rigorous solution' (section A.4 and A.5) which means that the targets are met.

5.2 Optional improvements

The following improvements on Focus + Farob + buckling prediction methods are foreseen:

- In Farob; calculate the sectional properties (e.g. bending stiffness) with laminate stiffness properties derived from the material properties of the fibres following **laminate theory**. Now the sectional properties are calculated by simply adding the Young's moduli.
- A more **detailed buckling load prediction of the shear web**, based on section B.5. When this is done well, the prediction covers collapse modes with short wave-length.
- Extend Farob with a model for the '**crushing loads**' in the shear webs.
- Include the contribution of **S-shaped** panel geometry on the buckling load.
- Complete the **rigorous solution** in routine *rlamod3* for non-uniform curvature, see A.5.
- Add an algorithm to the panel-based methods for the geometric **pre-buckling deformation** of the contour panels due to the so-called 'crushing loads'.
- Correct the program **Finstrip for sandwich layup** of which the facings have considerable individual bending stiffness and/or of which the stresses in the core contribute to the in-plane stiffness and bending stiffness. Also a proper contribution of the core to the bending stiffness can be an improvement of both the panel-based methods and Finstrip.
- In the framework of the Focus6 project, full featured (**thick shell**) finite element solvers will be implemented in Focus6. One of the options will be buckling analyses of rotor blades. This new feature will be used in addition to the existing buckling analyses tools of Focus.

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A STABILITY EQUATIONS OF CURVED PANELS

For the fibre-reinforced material of rotor blades the elastic stress-strain relations depend on the direction of loading. Rotor blade panels may have a sandwich construction in order to obtain a high out-of-plane bending stiffness in combination with a low weight. This appendix contains descriptions of:

- Laminate theory used to define the panel stiffness properties;
- Description of out-of-plane shear flexibility with a few parameters;
- Non-linear strain-displacement relations for curved panels;
- The linear stability equations for long orthotropic panels
  with non-uniform curvature, material, and loading;
- The non-linear equilibrium equations for long orthotropic panels
  with uniform material, uniform loading, and (weak) uniform curvature.

A.1 Shell stiffnesses

The stiffness relations that are derived here are based on the Kirchhoff-Love hypothesis:
"*A normal vector in the material of an unloaded shell remains straight and normal to the deformed shell reference plane after deformation*".
For strength analysis this means that the in-plane strain in any point in the shell is the sum of the strain of the shell reference plane and the distance with respect to this reference plane times the curvature from elastic deformation:

$$\begin{aligned} \epsilon_\mathrm{x}(z) &= \epsilon_\mathrm{x}(z_\mathrm{ref}) + (z - z_\mathrm{ref})\,\kappa_\mathrm{x} \\ \epsilon_\mathrm{y}(z) &= \epsilon_\mathrm{y}(z_\mathrm{ref}) + (z - z_\mathrm{ref})\,\kappa_\mathrm{y} \\ \gamma_\mathrm{xy}(z) &= \gamma_\mathrm{xy}(z_\mathrm{ref}) + (z - z_\mathrm{ref})\,\kappa_\mathrm{xy} \end{aligned} \tag{23}$$

The material stiffness relations of a layer have the general form:

$$\begin{pmatrix} \sigma_\mathrm{x} \\ \sigma_\mathrm{y} \\ \sigma_\mathrm{z} \\ \tau_\mathrm{yz} \\ \tau_\mathrm{xz} \\ \tau_\mathrm{xy} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_\mathrm{x} \\ \epsilon_\mathrm{y} \\ \epsilon_\mathrm{z} \\ \gamma_\mathrm{yz} \\ \gamma_\mathrm{xz} \\ \gamma_\mathrm{xy} \end{pmatrix} \tag{24}$$

Note that the matrix in this stiffness relation is symmetric.
For a material that is homogeneous and symmetric w.r.t. the $x-y$ plane this relation has the form

$$\begin{pmatrix} \sigma_\mathrm{x} \\ \sigma_\mathrm{y} \\ \sigma_\mathrm{z} \\ \tau_\mathrm{yz} \\ \tau_\mathrm{xz} \\ \tau_\mathrm{xy} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & s_{16} \\ s_{12} & s_{22} & s_{23} & 0 & 0 & s_{26} \\ s_{13} & s_{23} & s_{33} & 0 & 0 & s_{36} \\ 0 & 0 & 0 & s_{44} & s_{45} & 0 \\ 0 & 0 & 0 & s_{45} & s_{55} & 0 \\ s_{16} & s_{26} & s_{36} & 0 & 0 & s_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_\mathrm{x} \\ \epsilon_\mathrm{y} \\ \epsilon_\mathrm{z} \\ \gamma_\mathrm{yz} \\ \gamma_\mathrm{xz} \\ \gamma_\mathrm{xy} \end{pmatrix} \tag{25}$$

Following thin shell theory the out-of-plane normal stress σ_z is zero.
Elimination of $\epsilon_\mathrm{z} = -(s_{13}\,\epsilon_\mathrm{x} + s_{23}\,\epsilon_\mathrm{y} + s_{36}\,\gamma_\mathrm{xy})/s_{33}$ gives for the in-plane strains and stresses

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} s_{11} - s_{13}^2/s_{33} & s_{12} - s_{23}\, s_{13}/s_{33} & s_{16} - s_{36}\, s_{13}/s_{33} \\ s_{12} - s_{23}\, s_{13}/s_{33} & s_{22} - s_{23}^2/s_{33} & s_{26} - s_{23}\, s_{36}/s_{33} \\ s_{16} - s_{36}\, s_{13}/s_{33} & s_{26} - s_{23}\, s_{36}/s_{33} & s_{66} - s_{36}^2/s_{33} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix}$$

or in short form:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \tilde{s}_{16} \\ \tilde{s}_{12} & \tilde{s}_{22} & \tilde{s}_{26} \\ \tilde{s}_{16} & \tilde{s}_{26} & \tilde{s}_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{26}$$

This in-plane stiffness relation is still symmetric.

For most materials the elements of this in-plane stiffness matrix can be obtained either by calculation from the engineering constants of the fibres and resin, or by means of coupon tests.
For fibre-reinforced materials these orthotropic(!) stiffnesses are usually given with respect to the longitudinal '*l*' and transverse '*t*' fibre directions:

$$\begin{pmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & 0 \\ q_{12} & q_{22} & 0 \\ 0 & 0 & q_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_l \\ \epsilon_t \\ \gamma_{lt} \end{pmatrix} \tag{27}$$

For a layer of fibres of which the fibre direction *l* makes an angle ϕ with the '*x*' direction the stiffness relation after transformation is of the form (A.2).

The individual terms in this relation are related to the stiffnesses q_{11} etc. with respect to the fibre direction by (from Jones, [12] section 2.6 p.51):

$$\begin{aligned}
\tilde{s}_{11} &= q_{11}\,(\cos\phi)^4 + 2(q_{12} + 2\,q_{66})\,(\cos\phi)^2\,(\sin\phi)^2 + q_{22}\,(\sin\phi)^4 \ ; \\
\tilde{s}_{12} &= q_{12}\,(\,(\cos\phi)^4 + (\sin\phi)^4\,) + (q_{11} + q_{22} - 4\,q_{66})\,(\cos\phi)^2\,(\sin\phi)^2 \ ; \\
\tilde{s}_{22} &= q_{22}\,(\cos\phi)^4 + 2(q_{12} + 2\,q_{66})\,(\cos\phi)^2\,(\sin\phi)^2 + q_{11}\,(\sin\phi)^4 \ ; \\
\tilde{s}_{16} &= (q_{11} - q_{12} - 2\,q_{66})\,(\cos\phi)^3\,\sin\phi - (q_{22} - q_{12} - 2\,q_{66})\,(\sin\phi)^3\,\cos\phi \ ; \\
\tilde{s}_{26} &= (q_{11} - q_{12} - 2\,q_{66})\,(\sin\phi)^3\,\cos\phi - (q_{22} - q_{12} - 2\,q_{66})\,(\cos\phi)^3\,\sin\phi \ ; \\
\tilde{s}_{66} &= q_{66}\,(\,(\cos\phi)^4 + (\sin\phi)^4\,) + (q_{11} + q_{22} - 2(q_{12} + q_{66})\,)\,(\cos\phi)^2\,(\sin\phi)^2 \ .
\end{aligned} \tag{28}$$

The in-plane loading on the entire shell follows from thickness integration of the stresses:

$$\begin{aligned}
N_x &= \textstyle\int_{\text{thickness}\,z} (\tilde{s}_{11}(z)\,\epsilon_x(z) + \tilde{s}_{12}(z)\,\epsilon_y(z) + \tilde{s}_{16}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ ; \\
N_y &= \textstyle\int_{\text{thickness}\,z} (\tilde{s}_{12}(z)\,\epsilon_x(z) + \tilde{s}_{22}(z)\,\epsilon_y(z) + \tilde{s}_{26}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ ; \\
N_{xy} &= \textstyle\int_{\text{thickness}\,z} (\tilde{s}_{16}(z)\,\epsilon_x(z) + \tilde{s}_{26}(z)\,\epsilon_y(z) + \tilde{s}_{66}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ ;
\end{aligned}$$

while the out-of-plane bending moments follow from the integrations:

$$\begin{aligned}
M_x &= \textstyle\int_{\text{thickness}\,z} (z - z_{\text{ref}})\,(\tilde{s}_{11}(z)\,\epsilon_x(z) + \tilde{s}_{12}(z)\,\epsilon_y(z) + \tilde{s}_{16}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ ; \\
M_y &= \textstyle\int_{\text{thickness}\,z} (z - z_{\text{ref}})\,(\tilde{s}_{12}(z)\,\epsilon_x(z) + \tilde{s}_{22}(z)\,\epsilon_y(z) + \tilde{s}_{26}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ ; \\
M_{xy} &= \textstyle\int_{\text{thickness}\,z} (z - z_{\text{ref}})\,(\tilde{s}_{16}(z)\,\epsilon_x(z) + \tilde{s}_{26}(z)\,\epsilon_y(z) + \tilde{s}_{66}(z)\,\gamma_{xy}(z)\,)\ \mathrm{d}z \ .
\end{aligned}$$

Processing of the integrals gives the stiffness relations for an anisotropic shell

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \tag{29}$$

or in short form $\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$.

Here the terms of the matrices A , B , and D are

$$A_{ij} = \int_{\text{thickness}} \tilde{s}_{ij}(z)\,\mathrm{d}z \;;$$

$$B_{ij} = \int_{\text{thickness}} (z - z_{\text{ref}})\,\tilde{s}_{ij}(z)\,\mathrm{d}z \;;$$

$$D_{ij} = \int_{\text{thickness}} (z - z_{\text{ref}})^2\,\tilde{s}_{ij}(z)\,\mathrm{d}z \;. \tag{30}$$

Due to symmetry of the matrices and layer-stress only the terms for
$i\,j = 1\,1,\ 1\,2,\ 2\,2,\ 1\,6,\ 2\,6,$ and $6\,6$ have to be evaluated.

For layered shells of which the material is homogeneous through the thickness of each layer the integrals can be calculated by the summations

$$A_{ij} = \sum_{k=1}^{k=N_{\text{layers}}} (z_k - z_{k-1})\,\tilde{s}_{ij}(k) \;;$$

$$B_{ij} = \sum_{k=1}^{k=N_{\text{layers}}} \frac{1}{2}(z_k^2 - z_{k-1}^2 - 2(z_k - z_{k-1})z_{\text{ref}})\,\tilde{s}_{ij}(k) \;;$$

$$D_{ij} = \sum_{k=1}^{k=N_{\text{layers}}} \frac{1}{3}(\,(z_k - z_{\text{ref}})^3 - (z_{k-1} - z_{\text{ref}})^3\,)\,\tilde{s}_{ij}(k) \;.$$

Orthotropic panels

A special type of anisotropic panels are orthotropic panels for which the stiffness properties are symmetric with respect to the longitudinal- and transverse coordinate axes. This means that shear- and torsion effects are un-coupled from bending- and normal strain effects and is reflected in zero values for the stiffness matrix elements A_{16} , A_{26} , B_{16} , B_{26} , D_{16} , and D_{26} .

A.2 Out-of-plane shear deformation

Sandwich panels are layered shells characterised by one relatively flexible layer called "core", between two (laminates of) stiff layers called "facings". In fact all material layers have some out-of-plane shear flexibility although this is practically not significant except for the "core" layer that is designed to increase the overall panel thickness and to have little weight.

The out-of-plane shear deformation can be modelled in several ways. For sandwich panels with a relatively thick and soft core, a simple approach is to omit the contribution of the in-plane stiffness of the core itself and to assume that the facings do not have bending stiffness. In addition the sandwich panels are often treated as symmetric (having similar facings), which is usually true for the shear web in a rotor blade but which may not be true for sandwich panels in the blade contour.

This section first contains a description for the out-of-plane shear flexibility on basis of the fact that the transverse shear loading in a panel runs between the midplanes of the facings. The relation between the panel loading and the transverse shear deformation (-gradients) is given with stiffness matrices B_s and D_s, which are slightly smaller than the matrices B and D.

Next an approach is given in which the out-of-plane shear deformation and the out-of-plane shear stiffness is defined such that the influence of the out-of-plane shear gradients is described with the panel stiffness matrices B and D.

With shear loading modelled between facing midplanes

Under the assumption that the out-of-plane shear deformation in the core is uniform through the thickness and the core itself does not contribute significantly to the load-carrying capabilities of the panel the so-called "first-order shear theory" can be applied. This means that the expressions for the strains in the upper facing differ from those in the lower facing by an amount:

$$\epsilon_{\mathrm{x\,upper}}(z_u) = \epsilon_{\mathrm{x\,lower}}(z_l) + (z_u - z_l)\,\kappa_x - \gamma_{\mathrm{xz.core},x}\,t_c\ ;$$
$$\epsilon_{\mathrm{y\,upper}}(z_u) = \epsilon_{\mathrm{y\,lower}}(z_l) + (z_u - z_l)\,\kappa_y - \gamma_{\mathrm{yz.core},y}\,t_c\ ;$$
$$\gamma_{\mathrm{xy\,upper}}(z_u) = \gamma_{\mathrm{xy\,lower}}(z_l) + (z_u - z_l)\,\kappa_{xy} - (\gamma_{\mathrm{xz.core},y} + \gamma_{\mathrm{yz.core},x})\,t_c\ .$$

Here t_c is the core thickness. Because of the fact that the transverse shear stress can be calculated as if it runs between the facing mid-planes, it is more convenient to express the out-of-plane shear deformations γ_{xz} and γ_{yz} between the facing mid-planes, which are a distance h from each other.

Integration of the stiffnesses with these expressions for the local strains gives the sandwich-shell stiffness properties:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & B_{s11} & B_{s12} & B_{s16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{s12} & B_{s22} & B_{s26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & B_{s16} & B_{s26} & B_{s66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & D_{s11} & D_{s12} & D_{s16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & D_{s12} & D_{s22} & D_{s26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & D_{s16} & D_{s26} & D_{s66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz,x} \\ \gamma_{yz,y} \\ \gamma_{xz,y} + \gamma_{yz,x} \end{pmatrix}$$

or in short form
$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} + \begin{pmatrix} B_s \\ D_s \end{pmatrix} \cdot \begin{pmatrix} \gamma_{xz,x} \\ \gamma_{yz,y} \\ \gamma_{xz,y} + \gamma_{yz,x} \end{pmatrix} .$$

Here the matrix D_s contains the bending stiffness terms expressed as if they are due to in-plane strain variations of the facings only, which is related to the use of the core-flexibility as out-of-plane flexibility of the entire sandwich panel.

For sandwich panels of which the facings have membrane stiffness matrices that are proportional to each other, the reference plane for out-of-plane shear can be chosen such that the stiffness matrix B_s equals zero.

For sandwich panels of which the bending stiffness of the core and of the individual facings are negligible compared to the panel bending stiffnesses, the matrices B_s and D_s can be replaced by B and D for which the stiffness relations get the form:

$$\begin{pmatrix} N_\mathrm{x} \\ N_\mathrm{y} \\ N_\mathrm{xy} \\ M_\mathrm{x} \\ M_\mathrm{y} \\ M_\mathrm{xy} \end{pmatrix} = \begin{pmatrix} A & B \\ & \\ B & D \end{pmatrix} \cdot \begin{pmatrix} \epsilon_\mathrm{x} \\ \epsilon_\mathrm{y} \\ \gamma_\mathrm{xy} \\ \kappa_\mathrm{x} + \gamma_{\mathrm{xz},x} \\ \kappa_\mathrm{y} + \gamma_{\mathrm{yz},y} \\ \kappa_\mathrm{xy} + \gamma_{\mathrm{xz},y} + \gamma_{\mathrm{yz},x} \end{pmatrix} . \tag{31}$$

For the out-of-plane deformation expressed over a thickness h of the anisotropic elastic core with thickness t_c the additional relations are

$$\begin{pmatrix} N_\mathrm{yz} \\ N_\mathrm{xz} \end{pmatrix} = \begin{pmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{pmatrix} \cdot \begin{pmatrix} \gamma_\mathrm{yz} \\ \gamma_\mathrm{xz} \end{pmatrix} .$$

For an orthotropic core layer with its principal direction (G_{13} -dir) having an angle ϕ with the x axis, the stiffness properties are obtained from trigonometric decomposition of the out-of-plane shear deformation and -flexibility, which gives the inverse of the out-of-plane shear stiffness matrix:

$$(A^{-1})_{55} = (\,(\cos\phi)^2/G_{13} + (\sin\phi)^2/G_{23}\,)\; t_\mathrm{c}/h^2 \;;$$
$$(A^{-1})_{45} = \;\; \cos\phi\;\sin\phi\;(1/G_{13} - 1/G_{23})\; t_\mathrm{c}/h^2 \;;$$
$$(A^{-1})_{44} = (\,(\cos\phi)^2/G_{23} + (\sin\phi)^2/G_{13}\,)\; t_\mathrm{c}/h^2 \;.$$

The out-of-plane stiffness matrix elements follow from inverting the flexibility matrix A^{-1} :

$$A_{55} = \;\; (A^{-1})_{44}/(\,(A^{-1})_{55}\,(A^{-1})_{44} - {(A^{-1})_{45}}^2\,) \;;$$
$$A_{45} = -(A^{-1})_{45}/(\,(A^{-1})_{55}\,(A^{-1})_{44} - {(A^{-1})_{45}}^2\,) \;;$$
$$A_{44} = \;\; (A^{-1})_{55}/(\,(A^{-1})_{55}\,(A^{-1})_{44} - {(A^{-1})_{45}}^2\,) \;.$$

For implementation in a routine, it is recommended to use the flexibility matrix elements A^{-1} because they approach zero for laminates without typical sandwich layup.

Panels with a relatively small out-of-plane shear flexibility, $(A^{-1})_{44} << (b/\pi)^2/D_{\mathrm{s}\,22}$ and $(A^{-1})_{55} << (b/\pi)^2/D_{\mathrm{s}\,11}$ do have small out-of-plane shear deformations γ_xz and γ_yz .

A.3 Strain-displacement relations

The description of geometric instability is based among others on the non-linear strain-displacement relations. The form of these relations reflect the character and detail of the geometrical modelling.

The phenomenon of buckling involves the transition from in-plane strain energy to out-of-plane bending energy, which is the basis of some of the methods described in Appendix B. This means that for an accurate prediction of the buckling load, the description of the correct amount of in-plane strain energy (that is transformed into bending energy) is important. The second variation of the strain energy is calculated with the strain-displacement relations from the variation in shell deformation. Whether it concerns a simple panel method or a shell element of a finite element package, the strain-displacement relations that are used are of vital importance.

The complete set of expressions for the strains of shells with double curvature and twist that is also invariant for rigid body rotations are rather lengthy while it is also difficult to define an orthogonal coordinate system. For panels that have a small longitudinal curvature $1/R_{\mathrm{x}}$ and a small geometric twist $1/R_{\mathrm{xy}}$ but that may have a strong transverse curvature $1/R_{\mathrm{y}}$ the following strain-displacement relations are invariant for moderate rigid body rotations:

$$\begin{aligned}
\epsilon_{\mathrm{x}} &= u_{,x} - w/R_{\mathrm{x}} + \tfrac{1}{2}\, w_{,x}^{\,2} + \tfrac{1}{2}\, v_{,x}^{\,2} + \tfrac{1}{2}\, u_{,x}^{\,2}; \\
\epsilon_{\mathrm{y}} &= v_{,y} - w/R_{\mathrm{y}} + \tfrac{1}{2}\,(w_{,y} + v/R_{\mathrm{y}})^2 + \tfrac{1}{2}\,(v_{,y} - w/R_{\mathrm{y}})^2 + \tfrac{1}{2}\, u_{,y}^{\,2}\ ; \\
\gamma_{\mathrm{xy}} &= u_{,y} + v_{,x} - 2\,w/R_{\mathrm{xy}} + w_{,x}\,(w_{,y} + v/R_{\mathrm{y}}) + v_{,x}\,(v_{,y} - w/R_{\mathrm{y}}) + u_{,x}\,u_{,y}\ ; \\
\kappa_{\mathrm{x}} &= -w_{,xx}; \quad \kappa_{\mathrm{y}} = -w_{,yy} - v_{,y}/R_{\mathrm{y}}; \quad \kappa_{\mathrm{xy}} = -2\,w_{,xy} - (v_{,x} - u_{,y})/R_{\mathrm{y}}.
\end{aligned}$$

The expressions given here are for curvatures that describe a panel with its convex side toward the positive z direction. The x , y , and z axes form a right handed system. The quadratic terms in the derivatives of u and v for the in-plane strain are necessary to describe Euler buckling of long beams under axial compression.

With the fact that the collapse mode is dominated by out-of-plane deformations, the quadratic terms in u and in v can be omitted from the in-plane strains gives:

$$\begin{aligned}
\epsilon_{\mathrm{x}} &= u_{,x} - w/R_{\mathrm{x}} + \tfrac{1}{2}\, w_{,x}^{\,2}\ ; \\
\epsilon_{\mathrm{y}} &= v_{,y} - w/R_{\mathrm{y}} + \tfrac{1}{2}\,(w_{,y} + v/R_{\mathrm{y}})^2\ ; \\
\gamma_{\mathrm{xy}} &= u_{,y} + v_{,x} - 2\,w/R_{\mathrm{xy}} + w_{,x}\,(w_{,y} + v/R_{\mathrm{y}}) - v_{,x}\,w/R_{\mathrm{y}}\ .
\end{aligned}$$

With a further simplification of the expression for γ_{xy} and with a different expression for the elastic twist κ_{xy} gives the relations as reported by V.V. Novozhilov et al.:

$$\begin{aligned}
\epsilon_{\mathrm{x}} &= u_{,x} - w/R_{\mathrm{x}} + \tfrac{1}{2}\, w_{,x}^{\,2}\ ; \\
\epsilon_{\mathrm{y}} &= v_{,y} - w/R_{\mathrm{y}} + \tfrac{1}{2}\,(w_{,y} + v/R_{\mathrm{y}})^2\ ; \\
\gamma_{\mathrm{xy}} &= u_{,y} + v_{,x} - 2\,w/R_{\mathrm{xy}} + w_{,x}\,(w_{,y} + v/R_{\mathrm{y}})\ ; \\
\kappa_{\mathrm{x}} &= -w_{,xx}\ ; \qquad \kappa_{\mathrm{y}} = -w_{,yy} - v_{,y}/R_{\mathrm{y}}\ ; \qquad \kappa_{\mathrm{xy}} = -2\,w_{,xy} - 2\,v_{,x}/R_{\mathrm{y}}\ . \qquad (32)
\end{aligned}$$

These strain-displacement relations are still invariant for small rigid body rotations that are described with u , v , and w . The difference in the expression for the elastic twist was applied on basis of the assumption that the in-plane shear deformation times the transverse curvature ($(u_{,y} + v_{,x})/R_{\mathrm{y}}$) is small compared to the term $w_{,xy}$.

If also the transverse curvature $1/R_{\mathrm{y}}$ is small compared to the panel dimensions Donnell's "quasi shallow shell assumptions" are valid which gives the strain displacement relations in section A.6.

A.4 Stability equations for non-uniform long prismatic panels

In this section the stability equations are derived for long panels of a rotor blade that are prismatic along the blade axis (x direction) and that have a non-uniform geometry and material properties in transverse y direction. The latter direction runs along the airfoil contour. The z axis is defined such that it points 'inward', to the concave side of the panel if this has a positive curvature $1/R_{\mathrm{y}}$.

The equations are based on the fact that for marginal stability of an equilibrium state the second variation of the total energy for the (small amplitude) collapse mode of the loaded panel is zero.

The elastic energy in a loaded panel supported with an elastic stiffness k_{supp} is:

$$U_{\mathrm{E}} = \int_{y=0}^{y=b} \int_{x=0}^{x=L} \frac{1}{2} [\, N_{\mathrm{x}}\, \epsilon_{\mathrm{x}} + N_{\mathrm{y}}\, \epsilon_{\mathrm{y}} + N_{\mathrm{xy}}\, \gamma_{\mathrm{xy}} + Q_{\mathrm{x}}\, \gamma_{\mathrm{xz}} + Q_{\mathrm{yz}}\, \gamma_{\mathrm{yz}} + M_{\mathrm{x}}\, (\kappa_{\mathrm{x}} + \gamma_{\mathrm{xz},x})$$

$$+ M_{\mathrm{y}}\, (\kappa_{\mathrm{y}} + \gamma_{\mathrm{yz},y}) + M_{\mathrm{xy}}\, (\kappa_{\mathrm{xy}} + \gamma_{\mathrm{xz},y} + \gamma_{\mathrm{yz},x}) + k_{\mathrm{supp}}\, w^2 \,]\, \mathrm{d}x\, \mathrm{d}y \ .$$

The energy of the applied load is:

$$U_{\mathrm{L}} = \int_{y=0}^{y=b} \int_{x=0}^{x=L} \frac{1}{2} [\, -N_{\mathrm{x}}\, u_{,x} - N_{\mathrm{y}}\, v_{,y} - N_{\mathrm{xy}}\, (u_{,y} + v_{,x}) \,]\, \mathrm{d}x\, \mathrm{d}y \ .$$

For equilibrium, the variation of the energy due to variations in the deformation must be zero. At the load for bifurcation-buckling the state can have some variations in deformation pattern for which equilibrium is also satisfied. This (marginal) 'stability' of an equilibrium state is satisfied if the second variation of the strain energy and energy from the applied load is zero:
$\delta^2(U_{\mathrm{E}} + U_{\mathrm{L}}) = 0\,.$

The deformed state is described with a pre-buckling solution and a collapse mode:
$u = u_0 + \delta u\,,\ v = v_0 + \delta v$, and $w = w_0 + \delta w$ where the subscript $_0$ indicates the undeformed state (under loading) and the symbol δ indicates the variation that describes the collapse mode. Substitution of these terms in the strain-displacement relations gives:
$\epsilon_{\mathrm{x}} = \epsilon_{\mathrm{x}\,0} + \delta\epsilon_{\mathrm{x}} + \delta^2\epsilon_{\mathrm{x}}\,,\ \epsilon_{\mathrm{y}} = \epsilon_{\mathrm{y}\,0} + \delta\epsilon_{\mathrm{y}} + \delta^2\epsilon_{\mathrm{y}}$, and also $\kappa_{\mathrm{x}} = \kappa_{\mathrm{x}\,0} + \delta\kappa_{\mathrm{x}}$ etcetera,
and for the loads: $N_{\mathrm{x}} = N_{\mathrm{x}\,0} + \delta N_{\mathrm{x}} + \delta^2 N_{\mathrm{x}}\,,\ N_{\mathrm{y}} = N_{\mathrm{y}\,0} + \delta N_{\mathrm{y}} + \delta^2 N_{\mathrm{y}}$ etcetera.

The second variation of the energy from the applied loads equals zero: $\delta^2 U_{\mathrm{L}} = 0$ because the applied loading at the edges of the panel do not change during collapse while the edges itself remain on the same location for small collapse amplitudes at the bifurcation point.

The second variation of the elastic strain energy is:

$$\delta^2 U_{\mathrm{E}} = \int_{y=0}^{y=b} \int_{x=0}^{x=L} \frac{1}{2} [\ \delta^2 N_{\mathrm{x}}\, \epsilon_{\mathrm{x}\,0} + \delta N_{\mathrm{x}}\, \delta\epsilon_{\mathrm{x}} + N_{\mathrm{x}\,0}\, \delta^2\epsilon_{\mathrm{x}} + \delta^2 N_{\mathrm{y}}\, \epsilon_{\mathrm{y}\,0} + \delta N_{\mathrm{y}}\, \delta\epsilon_{\mathrm{y}}$$

$$+ N_{\mathrm{y}\,0}\, \delta^2\epsilon_{\mathrm{y}} + \delta^2 N_{\mathrm{xy}}\, \gamma_{\mathrm{xy}\,0} + \delta N_{\mathrm{xy}}\, \delta\gamma_{\mathrm{xy}} + N_{\mathrm{xy}\,0}\, \delta^2\gamma_{\mathrm{xy}} + \delta Q_{\mathrm{xz}}\, \delta\gamma_{\mathrm{xz}}$$

$$+ \delta Q_{\mathrm{yz}}\, \delta\gamma_{\mathrm{yz}} + \delta M_{\mathrm{x}}\, (\delta\kappa_{\mathrm{x}} + \delta\gamma_{\mathrm{xz},x}) + \delta M_{\mathrm{y}}\, (\delta\kappa_{\mathrm{y}} + \delta\gamma_{\mathrm{yz},y})$$

$$+ \delta M_{\mathrm{xy}}\, (\delta\kappa_{\mathrm{xy}} + \delta\gamma_{\mathrm{xz},y} + \delta\gamma_{\mathrm{yz},x}) + k_{\mathrm{supp}}\, (\delta w)^2 \ ]\, \mathrm{d}x\, \mathrm{d}y \ = \ 0 \ .$$

Based on the assumptions that $\kappa_{\mathrm{x}\,0}$ = 0, $\kappa_{\mathrm{y}\,0}$ = 0, $\kappa_{\mathrm{xy}\,0}$ = 0, $\gamma_{\mathrm{xz}\,0}$ = 0, and $\gamma_{\mathrm{yz}\,0}$ = 0 and the relation

$$\begin{pmatrix} \delta^2 N_{\mathrm{x}} \\ \delta^2 N_{\mathrm{y}} \\ \delta^2 N_{\mathrm{xy}} \end{pmatrix} = A \begin{pmatrix} \delta^2 \epsilon_{\mathrm{x}} \\ \delta^2 \epsilon_{\mathrm{y}} \\ \delta^2 \gamma_{\mathrm{xy}} \end{pmatrix}$$

one can derive that $N_{\mathrm{x}\,0}\, \delta^2\epsilon_{\mathrm{x}} + N_{\mathrm{y}\,0}\, \delta^2\epsilon_{\mathrm{y}} + N_{\mathrm{xy}\,0}\, \delta^2\gamma_{\mathrm{xy}} = \delta^2 N_{\mathrm{x}}\, \epsilon_{\mathrm{x}\,0} + \delta^2 N_{\mathrm{y}}\, \epsilon_{\mathrm{y}\,0} + \delta^2 N_{\mathrm{xy}}\, \gamma_{\mathrm{xy}\,0}$.

Using this relation to replace the terms with second variations of the in-plane forces in the energy expression, and substitution of the stiffness relations gives:

$$\delta^2 U_{\rm E} = \int_{y=0}^{y=b} \int_{x=0}^{x=L} [\, \frac{1}{2}\, N_{\rm x\,0}\, (w_{,x})^2 + \frac{1}{2}\, N_{\rm y\,0}\, (w_{,y} + v/R_{\rm y})^2 + N_{\rm xy\,0}\, w_{,x}\, (w_{,y} + v/R_{\rm y})$$

$$+ \frac{1}{2} (\epsilon_{\rm x} \quad \epsilon_{\rm y} \quad \gamma_{\rm xy}) A \begin{pmatrix} \epsilon_{\rm x} \\ \epsilon_{\rm y} \\ \gamma_{\rm xy} \end{pmatrix} + (\epsilon_{\rm x} \quad \epsilon_{\rm y} \quad \gamma_{\rm xy}) B \begin{pmatrix} \kappa_{\rm x} + \gamma_{{\rm xz},x} \\ \kappa_{\rm y} + \gamma_{{\rm yz},y} \\ \kappa_{\rm xy} + \gamma_{{\rm xz},y} + \gamma_{{\rm yz},x} \end{pmatrix}$$

$$+ \frac{1}{2}\, (\kappa_{\rm x} + \gamma_{{\rm xz},x} \quad \kappa_{\rm y} + \gamma_{{\rm yz},y} \quad \kappa_{\rm xy} + \gamma_{{\rm xz},y} + \gamma_{{\rm yz},x}) D \begin{pmatrix} \kappa_{\rm x} + \gamma_{{\rm xz},x} \\ \kappa_{\rm y} + \gamma_{{\rm yz},y} \\ \kappa_{\rm xy} + \gamma_{{\rm xz},y} + \gamma_{{\rm yz},x} \end{pmatrix}$$

$$+ \frac{1}{2}\, A_{55}\, {\gamma_{\rm xz}}^2 + A_{45}\, \gamma_{\rm xz}\, \gamma_{\rm yz} + \frac{1}{2}\, A_{44}\, {\gamma_{\rm yz}}^2 + \frac{1}{2}\, k_{\rm supp}\, w^2 \,]\, {\rm d}x\, {\rm d}y \;=\; 0 \;. \qquad (33)$$

For simplicity the symbols δ for all first-order variations of the degrees of freedom are omitted in this expression, while the subscripts $_0$ still indicate the steady-state of the loaded panel.

If the strain-displacement relations are substituted, the general form of this expression is:

$$\int_y \int_x F(x, y, u_{,x}, u_{,y}, v, v_{,x}, v_{,y}, w, w_{,x}, w_{,y}, w_{,xx}, w_{,xy}, w_{,yy}, \gamma_{\rm xz}, \gamma_{{\rm xz},x}, \gamma_{{\rm xz},y}, \gamma_{\rm yz}, \gamma_{{\rm yz},x}, \gamma_{{\rm yz},y})\, {\rm d}x\, {\rm d}y = 0 \,.$$

From this integral equation a set of stability equations can be derived using Trefftz's criterion for each of the deformations: u, v, w, $\gamma_{\rm xz}$, and $\gamma_{\rm yz}$, see also Appendix A of [5].

For the u direction the stability equation is given by: $-\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_{,x}}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_{,y}}) = 0$.
Performing the derivatives and regrouping of some terms gives the stability equation:

$$N_{{\rm x},x} + N_{{\rm xy},y} \;=\; 0 \;. \qquad (34)$$

The stability equation for the v direction follows from: $\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial v_{,x}}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial v_{,y}}) = 0$.
Performing the derivatives and regrouping of some terms gives the stability equation:

$$N_{\rm y\,0}\, (w_{,y} + \frac{v}{R_{\rm y}})/R_{\rm y} + N_{\rm xy\,0}\; w_{,x}/R_{\rm y} - N_{{\rm xy},x} - N_{{\rm y},y} + 2\, \frac{\partial}{\partial x}(\frac{M_{\rm xy}}{R_{\rm y}}) + \frac{\partial}{\partial y}(\frac{M_{\rm y}}{R_{\rm y}}) \;=\; 0 \;. \qquad (35)$$

The stability equation for the w direction follows from:
$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial w_{,x}}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial w_{,y}}) + \frac{\partial^2}{(\partial x)^2}(\frac{\partial F}{\partial w_{,xx}}) + \frac{\partial^2}{\partial y\, \partial x}(\frac{\partial F}{\partial w_{,xy}}) + \frac{\partial^2}{(\partial y)^2}(\frac{\partial F}{\partial w_{,yy}}) = 0$.
Performing the derivatives and regrouping of some terms gives the stability equation:

$$\frac{\partial}{\partial x}(N_{\rm x\,0}\, w_{,x}) + \frac{\partial}{\partial y}(N_{\rm y\,0}\, (w_{,y} + \frac{v}{R_{\rm y}})) + \frac{\partial}{\partial x}(N_{\rm xy\,0}\, (w_{,y} + \frac{v}{R_{\rm y}})) + \frac{\partial}{\partial y}(N_{\rm xy\,0}\, w_{,x})$$

$$+ N_{\rm x}/R_{\rm x} + N_{\rm y}/R_{\rm y} + 2\, N_{\rm xy}/R_{\rm xy} + M_{{\rm x},xx} + M_{{\rm y},yy} + 2\, M_{{\rm xy},xy} - k_{\rm supp}\, w \;=\; 0 \;. \qquad (36)$$

The stability equation for the $\gamma_{\rm xz}$ direction follows from: $\frac{\partial F}{\partial \gamma_{\rm xz}} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial \gamma_{{\rm xz},x}}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial \gamma_{{\rm xz},y}}) = 0$.
Performing the derivatives and regrouping of some terms gives the stability equation:

$$Q_{\rm xz} - M_{{\rm x},x} - M_{{\rm xy},y} \;=\; 0 \;. \qquad (37)$$

The $5^{\rm th}$ stability equation for the $\gamma_{\rm yz}$ direction follows from: $\frac{\partial F}{\partial \gamma_{\rm yz}} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial \gamma_{{\rm yz},x}}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial \gamma_{{\rm yz},y}}) = 0$.
Performing the derivatives and regrouping of some terms gives the stability equation:

$$Q_{\rm yz} - M_{{\rm xy},x} - M_{{\rm y},y} \;=\; 0 \;. \qquad (38)$$

Equations for long prismatic panels of orthotropic material

Except for the blade root area, the panels of a rotor blade do have a quasi-prismatic geometry, loading, and material distribution. For this part of the rotor blade the collapse mode is sinusoidal in spanwise (x) direction which leaves a set of equations in the transverse coordinate y. In this section, this set of equations in y will be derived as a set of first-order linear differential equations from the 5 stability equations in the previous section, from the stiffness relations, and from the expressions for the strain variations.

In addition to the 5 stability equations (34), (35), (36), (37), and (38) one has the 8 stiffness relations (6 for conventional panels and 2 for the sandwich flexibilities) and the 3 in-plane strain displacement relations.

For orthotropic panels the stiffness relations can be reduced to:

$$\begin{pmatrix} N_{\rm x} \\ N_{\rm y} \\ M_{\rm x} \\ M_{\rm y} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{12} & A_{22} & B_{12} & B_{22} \\ B_{11} & B_{12} & D_{11} & D_{12} \\ B_{12} & B_{22} & D_{12} & D_{22} \end{pmatrix} \cdot \begin{pmatrix} u_{,x} - w/R_{\rm x} \\ v_{,y} - w/R_{\rm y} \\ -w_{,xx} + \gamma_{{\rm xz},x} \\ \beta_{{\rm y},y} \end{pmatrix} \; ;$$

$$\begin{pmatrix} N_{\rm xy} \\ M_{\rm xy} \end{pmatrix} = \begin{pmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{pmatrix} \cdot \begin{pmatrix} u_{,y} + v_{,x} \\ 2\,\beta_{{\rm y},x} + \gamma_{{\rm xz},y} - \gamma_{{\rm yz},x} \end{pmatrix} \; ;$$

and the sandwich stiffness relations $Q_{\rm xz} = A_{55}\,\gamma_{\rm xz}$ and $Q_{\rm yz} = A_{44}\,\gamma_{\rm yz}$.

The panel stiffness relations can be rewritten (by inverting partially or completely) to:

$$\begin{pmatrix} u_{,y} + v_{,x} \\ 2\,\beta_{{\rm y},x} + \gamma_{{\rm xz},y} - \gamma_{{\rm yz},x} \end{pmatrix} = \frac{1}{A_{66}\,D_{66} - B_{66}^2} \begin{pmatrix} D_{66} & -B_{66} \\ -B_{66} & A_{66} \end{pmatrix} \cdot \begin{pmatrix} N_{\rm xy} \\ M_{\rm xy} \end{pmatrix} \; .$$

These expressions for the in-plane strains will be used while the rotation of the normal vector of the panel material is introduced $\beta_{\rm y} = -w_{,y} - v/R_{\rm y} + \gamma_{\rm yz}$.

After some manipulations a set of 10 first order differential equations in y is obtained in the 13 variables: u, v, w, $\beta_{\rm y}$, $\gamma_{\rm xz}$, $N_{\rm x}$, $N^*_{\rm y}$, $N_{\rm xy}$, $Q_{\rm xz}$, $Q_{\rm yz}$, $M_{\rm x}$, $M_{\rm y}$, and $M_{\rm xy}$ with $N^*_{\rm y} = N_{\rm y} - M_{\rm y}/R_{\rm y}$.

$$\begin{aligned}
N_{{\rm xy},y} &= -\,N_{{\rm x},x} \; ; \\
N^*_{{\rm y}\,,y} &= -\,N_{{\rm xy},x} + 2\,M_{{\rm xy},x}\,/R_{\rm y} \; ; \\
Q_{{\rm yz},y} &= -\,N_{{\rm x}\,0}\,w_{,xx} - Q_{{\rm xz},x} - N_{\rm x}/R_{\rm x} - N^*_{\rm y}/R_{\rm y} - M_{\rm y}/R_{\rm y}^{\,2} + k_{\rm supp}\,w \; ; \\
M_{{\rm y},y} &= -\,M_{{\rm xy},x} + Q_{\rm yz} \; ; \\
M_{{\rm xy},y} &= -\,M_{{\rm x},x} + Q_{\rm xz} \; ; \\
w_{,y} &= -\,\beta_{\rm y} - v/R_{\rm y} + Q_{\rm yz}/A_{44} \; ; \\
\beta_{{\rm y},y} &= -\,(B_{22}/det22)\,(N^*_{\rm y} + M_{\rm y}/R_{\rm y}) + (A_{22}/det22)\,M_{\rm y} \\
&\qquad - S_{21}\,(u_{,x} - w/R_{\rm x}) - S_{22}\,(-w_{,xx} + \gamma_{{\rm xz},x}) \; ; \\
v_{,y} &= w/R_{\rm y} + (D_{22}/det22)\,(N^*_{\rm y} + M_{\rm y}/R_{\rm y}) - (B_{22}/det22)\,M_{\rm y} \\
&\qquad - S_{11}\,(u_{,x} - w/R_{\rm x}) - S_{12}\,(-w_{,xx} + \gamma_{{\rm xz},x}) \; ; \\
u_{,y} &= -\,v_{,x} + (D_{66}/det66)\,N_{\rm xy} - (B_{66}/det66)\,M_{\rm xy} \; ; \\
\gamma_{{\rm xz},y} &= Q_{{\rm yz},x}/A_{44} - 2\,\beta_{{\rm y},x} - (B_{66}/det66)\,N_{\rm xy} + (A_{66}/det66)\,M_{\rm xy} \; .
\end{aligned}$$

Here the terms $det22$ and $det66$ stand for $det22 = A_{22}\,D_{22} - B_{22}^2$ and $det66 = A_{66}\,D_{66} - B_{66}^2$.
The matrix elements S_{ij} stand for:

$S_{11} = (D_{22}\,A_{12} - B_{12}\,B_{22})/det22$;
$S_{12} = (D_{22}\,B_{12} - D_{12}\,B_{22})/det22$;
$S_{21} = (A_{22}\,B_{12} - A_{12}\,B_{22})/det22$;
$S_{22} = (A_{22}\,D_{12} - B_{12}\,B_{22})/det22$.

In addition to these 10 first-order differential equations (preceding page) one has the three expressions from the (partially inverted-) stiffness relations:

$$N_{\mathrm{x}} = S_{11}\,N^*_{\mathrm{y}} + (S_{11}/R_{\mathrm{y}} + S_{21})\,M_{\mathrm{y}} + T_{11}\,(u_{,x} - w/R_{\mathrm{x}}) + T_{12}\,(-w_{,xx} + \gamma_{\mathrm{xz},x})\ ;$$
$$M_{\mathrm{x}} = S_{12}\,N^*_{\mathrm{y}} + (S_{12}/R_{\mathrm{y}} + S_{22})\,M_{\mathrm{y}} + T_{12}\,(u_{,x} - w/R_{\mathrm{x}}) + T_{22}\,(-w_{,xx} + \gamma_{\mathrm{xz},x})\ ;$$
$$\gamma_{\mathrm{xz}} = Q_{\mathrm{xz}}/A_{55}\ .$$

The last expression will be used to eliminate γ_{xz}.

Here the matrix elements T_{ij} stand for:

$T_{11} = A_{11} - A_{12}\,S_{11} - B_{12}\,S_{21}$;
$T_{12} = B_{11} - A_{12}\,S_{12} - B_{12}\,S_{22}$;
$T_{22} = D_{11} - B_{12}\,S_{12} - D_{12}\,S_{22}$.

Substitution of the expressions for N_{x}, M_{x}, and γ_{xz} gives the 10 equations in 10 variables:

$$\begin{aligned}
N_{\mathrm{xy},y} &= -\,S_{11}\,N^*_{\mathrm{y},x} - (S_{11}/R_{\mathrm{y}} + S_{21})\,M_{\mathrm{y},x} - T_{11}\,u_{,xx} \\
&\quad + T_{11}\,w_{,x}/R_{\mathrm{x}} + T_{12}\,w_{,xxx} - T_{12}\,Q_{\mathrm{xz},xx}/A_{55}\ ; \\
N^*_{\mathrm{y}\,,y} &= -\,N_{\mathrm{xy},x} + 2\,M_{\mathrm{xy},x}/R_{\mathrm{y}}\ ; \\
Q_{\mathrm{yz},y} &= -\,N_{\mathrm{x}\,0}\,w_{,xx} - Q_{\mathrm{xz},x} - T_{12}\,(Q_{\mathrm{xz},x}/A_{55})/R_{\mathrm{x}} - N^*_{\mathrm{y}}/R_{\mathrm{y}} \\
&\quad - S_{11}\,N^*_{\mathrm{y}}/R_{\mathrm{x}} - M_{\mathrm{y}}/R_{\mathrm{y}}^2 - (S_{11}/R_{\mathrm{y}} + S_{21})\,M_{\mathrm{y}}/R_{\mathrm{x}} \\
&\quad - T_{11}\,u_{,x}/R_{\mathrm{x}} + T_{11}\,w/R_{\mathrm{x}}^2 + T_{12}\,w_{,xx}/R_{\mathrm{x}} + k_{\mathrm{supp}}\,w\ ; \\
M_{\mathrm{y},y} &= Q_{\mathrm{yz}} - M_{\mathrm{xy},x}\ ; \\
M_{\mathrm{xy},y} &= -\,S_{12}\,N^*_{\mathrm{y},x} - (S_{12}/R_{\mathrm{y}} + S_{22})\,M_{\mathrm{y},x} - T_{12}\,u_{,xx} \\
&\quad + T_{12}\,w_{,x}/R_{\mathrm{x}} + T_{22}\,w_{,xxx} - T_{22}\,Q_{\mathrm{xz},xx}/A_{55} + Q_{\mathrm{xz}}\ ; \\
w_{,y} &= -\,\beta_{\mathrm{y}} - v/R_{\mathrm{y}} + Q_{\mathrm{yz}}/A_{44}\ ; \\
\beta_{\mathrm{y},y} &= -\,(B_{22}/det22)\,N^*_{\mathrm{y}} + M_{\mathrm{y}}\,(A_{22} - B_{22}/R_{\mathrm{y}})/det22 \\
&\quad - S_{21}\,(u_{,x} - w/R_{\mathrm{x}}) - S_{22}\,(-w_{,xx} + Q_{\mathrm{xz},x}/A_{55})\ ; \\
v_{,y} &= w/R_{\mathrm{y}} + (D_{22}/det22)\,N^*_{\mathrm{y}} + M_{\mathrm{y}}\,(D_{22}/R_{\mathrm{y}} - B_{22})/det22 \\
&\quad - S_{11}\,(u_{,x} - w/R_{\mathrm{x}}) - S_{12}\,(-w_{,xx} + Q_{\mathrm{xz},x}/A_{55})\ ; \\
u_{,y} &= -\,v_{,x} + (D_{66}/det66)\,N_{\mathrm{xy}} - (B_{66}/det66)\,M_{\mathrm{xy}}\ ; \\
(Q_{\mathrm{xz}}/A_{55})_{,y} &= Q_{\mathrm{yz},x}/A_{44} - 2\,\beta_{\mathrm{y},x} - (B_{66}/det66)\,N_{\mathrm{xy}} + (A_{66}/det66)\,M_{\mathrm{xy}}\ .
\end{aligned}$$

The solution of this set of equations is a sinusoidal function in longitudinal direction $\sin(\pi\,x/L) = \sin(\mu_{\mathrm{x}}\,x)$ for the variables w, v, β_{y}, N^*_{y}, Q_{yz}, and M_{y}.

From the order of the *x* derivatives one can derive that the following variables have a cosine distribution $\cos(\mu_{\mathrm{x}}\,x)$: u, N_{xy}, Q_{xz}, and M_{xy}.

Substitution of these spanwise (x-) distributions, applying the x- derivatives, and omitting the $\sin(\mu_\mathrm{x}\, x)$ and $\cos(\mu_\mathrm{x}\, x)$ functions gives the equations (after some re-ordering):

$$
\begin{aligned}
Q_{\mathrm{yz},y} &= \mu_\mathrm{x}^2\, N_{\mathrm{x}0}\, w - (1/R_\mathrm{y} + S_{11}/R_\mathrm{x}) N_\mathrm{y}^* - (1/R_\mathrm{y}^2 + (S_{11}/R_\mathrm{y} + S_{21})/R_\mathrm{x}) M_\mathrm{y} \\
&\quad + (T_{11}/R_\mathrm{x}^2 - \mu_\mathrm{x}^2\, T_{12}/R_\mathrm{x} + k_\mathrm{supp})\, w + \mu_\mathrm{x}(T_{11}/R_\mathrm{x}) u + \mu_\mathrm{x}(1 + (T_{12}/R_\mathrm{x})/A_{55})\, Q_\mathrm{xz} \;; \\
w_{,y} &= -\,\beta_\mathrm{y} - v/R_\mathrm{y} + Q_\mathrm{yz}/A_{44} \;; \\
Q_{\mathrm{xz},y}/A_{55} &= \mu_\mathrm{x}\, Q_\mathrm{yz}/A_{44} - 2\,\mu_\mathrm{x}\,\beta_\mathrm{y} - (B_{66}/det66)\, N_\mathrm{xy} + (A_{66}/det66)\, M_\mathrm{xy} \;; \\
M_{\mathrm{xy},y} &= -\,\mu_\mathrm{x}\, S_{12}\, N_\mathrm{y}^* - \mu_\mathrm{x}\,(S_{12}/R_\mathrm{y} + S_{22})\, M_\mathrm{y} + \mu_\mathrm{x}^2\, T_{12}\, u \\
&\quad + \mu_\mathrm{x}\,(T_{12}/R_\mathrm{x} - \mu_\mathrm{x}^2\, T_{22})\, w + (1 + \mu_\mathrm{x}^2\, T_{22}/A_{55})\, Q_\mathrm{xz} \;; \\
M_{\mathrm{y},y} &= Q_\mathrm{yz} + \mu_\mathrm{x}\, M_\mathrm{xy} \;; \\
\beta_{\mathrm{y},y} &= -\,(B_{22}/det22)\, N_\mathrm{y}^* + ((A_{22} - B_{22}/R_\mathrm{y})/det22)\, M_\mathrm{y} \\
&\quad + \mu_\mathrm{x}\, S_{21}\, u + (S_{21}/R_\mathrm{x} - \mu_\mathrm{x}^2\, S_{22})\, w + \mu_\mathrm{x}\, S_{22}\, Q_\mathrm{xz}/A_{55} \;; \\
v_{,y} &= (D_{22}/det22)\, N_\mathrm{y}^* + ((D_{22}/R_\mathrm{y} - B_{22})/det22)\, M_\mathrm{y} \\
&\quad + \mu_\mathrm{x}\, S_{11}\, u + (1/R_\mathrm{y} + S_{11}/R_\mathrm{x} - \mu_\mathrm{x}^2\, S_{12})\, w + \mu_\mathrm{x}\, S_{12}\, Q_\mathrm{xz}/A_{55} \;; \\
N_{\mathrm{y}}^*{}_{,y} &= \mu_\mathrm{x}\, N_\mathrm{xy} - 2\,\mu_\mathrm{x}\, M_\mathrm{xy}/R_\mathrm{y} \;; \\
u_{,y} &= -\,\mu_\mathrm{x}\, v + (D_{66}/det66)\, N_\mathrm{xy} - (B_{66}/det66)\, M_\mathrm{xy} \;; \\
N_{\mathrm{xy},y} &= -\,\mu_\mathrm{x}\, S_{11}\, N_\mathrm{y}^* - \mu_\mathrm{x}\,(S_{11}/R_\mathrm{y} + S_{21})\, M_\mathrm{y} + \mu_\mathrm{x}^2\, T_{11}\, u \\
&\quad + \mu_\mathrm{x}\,(T_{11}/R_\mathrm{x} - \mu_\mathrm{x}^2\, T_{12})\, w + \mu_\mathrm{x}^2\, T_{12}\, Q_\mathrm{xz}/A_{55} \,. \qquad (39)
\end{aligned}
$$

If the set of equations (39) would be of the form: $\partial\vec{f}/\partial y = \bar{U}\cdot\vec{f}$ then they could be integrated straightforward (e.g. using the Euler method) over the panel width. However, if the layup does not have out-of-plane shear flexibility (if $1/A_{55} = 0$) the third equation does not give an expression for $Q_{\mathrm{xz},y}$. To deal with this fact, the derivatives in this set of equations are evaluated with finite differences over each Δy interval: $\partial\vec{f}/\partial y = (\vec{f}_{i+1} - \vec{f}_i)/\Delta y = \Delta\vec{f}/\Delta y$.
Here the index $_i$ indicates the values for the subsequent y coordinates.

Likewise the right-hand side terms are evaluated with the average values between the begin and end of each interval Δy: $(\vec{f}_{i+1} + \vec{f}_i)/2 = \vec{f}_i + \Delta\vec{f}/2$. For the routines that are used under Farob (such as *rlamod3*) the material properties are constant for each Δy interval.

The resulting set of equations in $\Delta\vec{f}$ is:

$$(\bar{I}^* - (\Delta y/2)\,\bar{U})\cdot\Delta\vec{f} = \Delta y\,\bar{U}\cdot\vec{f}_i \,. \qquad (40)$$

Except for the third element I^* is a diagonal unit-matrix. Even with the fact that $1/A_{55}$ (third diagonal element) may be zero, a solution can obtain an explicit expression by (partial) decomposition of the matrix $(\bar{I}^* - \bar{U}\,\Delta y/2)$. Here one can use the fact that matrix $\bar{U}$ is sparse to reduce the amount of CPU needed.

Discontinuities of variables from non-uniform material distribution

The set of first order linear differential equations (39) are formulated for constant material properties. With the fact that in the routines (that are called by Farob) the material properties are constant within each interval Δy the equations (39) can be used to integrate the degrees of freedom that describe the collapse mode. At the 'nodes' between the adjacent intervals Δy the material properties (and axial loading N_{x0}) may have discontinuities. These discontinuities may also appear for some of the degrees of freedom such as Q_{xz} and γ_{xz}.

Reduced solution for flat plates

For flat plates ($1/R_x = 0$ and $1/R_y = 0$) the 6 differential equations for Q_{yz}, w, Q_{xz}, M_{yz}, M_y, and β_y can be written in the form:

$$Q_{yz,y} = \mu_x^2 N_{x0}\, w + \mu_x Q_{xz} \; ;$$

$$w_{,y} = -\beta_y + Q_{yz}/A_{44} \; ;$$

$$Q_{xz,y}/A_{55} = \mu_x Q_{yz}/A_{44} - 2\mu_x \beta_y + (A_{66}/det66)\, M_{xy} \; ;$$

$$M_{xy,y} = -\mu_x S_{22} M_y + \mu_x^2 T_{12}\, u - \mu_x^3 T_{22}\, w + (1 + \mu_x^2 T_{22}/A_{55})\, Q_{xz} \; ;$$

$$M_{y,y} = Q_{yz} + \mu_x M_{xy} \; ;$$

$$\beta_{y,y} = (A_{22}/det22)\, M_y + \mu_x S_{21}\, u - \mu_x^2 S_{22}\, w + \mu_x S_{22} Q_{xz}/A_{55} \; . \qquad (41)$$

This set of 6 equations still depend on the in-plane displacement variation u if S_{21} and T_{12} are non-zero, which is for non-symmetric laminates. A set of 6 equations in 6 unknowns is realised for symmetric panels (zero B matrix) or if the contribution of the B matrix is included in the so-called 'reduced stiffness matrix' $\tilde{D}$, see also section A.6. Omission of the terms of the B matrix elements in the expressions for S_{ij} and T_{ij} and using $\tilde{D}$ instead of D gives:

$S_{12} = S_{21} = T_{12} = 0$;
$S_{22} = \tilde{D}_{12}/\tilde{D}_{22}$;
$T_{22} = \tilde{D}_{11} - \tilde{D}_{12}^2/\tilde{D}_{22}$;
$A_{22}/det22 = 1/\tilde{D}_{22}$;
$A_{66}/det66 = 1/\tilde{D}_{66}$.

Initially a solution for this simplified set of equations was implemented for 'method 3'.

A.5 Rigorous solution of the 2-point boundary value problem

Based on the (adjacent) equilibrium equations derived in the previous section a rigorous solution method is implemented in routine *rlamod3* that is linked under Focus, see section D.3. This method is called 'rigorous solution' because it accounts for the detailed description over the panel width of panel curvature, axial load distribution, and material properties (including sandwich).

The deformation pattern solved in this routine can have any arbitrary shape in transverse direction while it has a sinusoidal function in longitudinal direction (separation of variables). This description implies that all asymmetric effects such as shear loading, geometric twist, and anisotropic material properties are not included. The effect of longitudinal curvature however is still included.

Solution method

The solution method for contour panel buckling solves the collapse mode as a two-point boundary value problem. The complete set of stability equations for this 2-point boundary value problem includes 10 first-order equations such as derived in the previous section.

The solution method that was initially implemented was for a reduced set of 6 differential equations for the case of flat panels, although still with non-uniform material properties. This smaller set of equations reduces the complexity and the risk of over- or under-flow.

Edge constraints

The remaining 2-point boundary value problem was solved for the edge constraints:

1 the out-of-plane displacements w are zero;
2 the bending moment variations M_{y} are zero;
3 the longitudinal out-of-plane shear loads Q_{xz} are zero.

The first two edge constraints describe so-called 'simply supported' edges and imply an under-estimation compared with a real blade.

For curved panels buckling also involves in-plane load variations and deformations, so that additional conditions have to be assessed for u with N_{xy} and also for v with N_{y}. The different combination of edge constrains are indicated in literature with SS-1, SS-2, SS-3, or SS-4. Unfortunately not all authors use the same notation. In this report $SS-4$ is used if both in-plane deformations u and v are assumed zero. For sandwich panels a fifth edge constraint has to be considered for γ_{yz}.

Panels of a rotor blade structure are usually connected with some amount of shear/torsion/twist stiffness due to the lap-joints. These joints are mostly of +/- 45°layers. This means that one may consider the in-plane displacement variations (along the edge) of the collapse mode u to be zero at the edges. The transverse displacement variations v of the collapse mode may have some flexibility, especially if the panel-edge is also an edge of the cross-section, such as the trailing edge. For this variable it is assumed that $N_{\mathrm{y}} = 0$. This set of edge constrains is called SS-3.

Solution process

A non-zero solution of the linear set of first order differential equations is obtained if the equations are satisfied and if also the variables apply to the edge constraints (see next sub-section). This will be the case for a specific value of the applied axial loading $N_{\mathrm{x}\,0}$. With the fact that the axial loading varies over the panel width y the critical loading will be expressed as factor λ on the applied load level $N_{\mathrm{x}\,0}$. The solution process is in fact a search process in which the load factor λ is increased until the complete 2-point boundary value problem is satisfied.

A.6 Donnell's non-linear equations for imperfect shells

For thin-walled shells of which the dimensions and the radii of curvature are large with respect to the wall thickness Donnell has formulated the 'quasi shallow shell assumptions':

1 The elastic curvatures of the shell do not depend on the in-plane deformations;

2 In the in-plane equilibrium equations the product of the out-of-plane shear loading and the curvature of the panel can be neglected with respect to the gradients of the in-plane loading.

With these assumptions the strain-displacement relations in (32) reduce to the simple form:

$$\begin{aligned}
\epsilon_{\mathrm{x}} &= u_{,x} - w/R_{\mathrm{x}} + \tfrac{1}{2}\, w_{,x}^{\;2} \;; \\
\epsilon_{\mathrm{y}} &= v_{,y} - w/R_{\mathrm{y}} + \tfrac{1}{2}\, w_{,y}^{\;2} \;; \\
\gamma_{\mathrm{xy}} &= u_{,y} + v_{,x} - 2\, w/R_{\mathrm{xy}} + w_{,x}\, w_{,y} \;; \\
\kappa_{\mathrm{x}} &= -w_{,xx} \;; \qquad \kappa_{\mathrm{y}} = -w_{,yy} \;; \qquad \kappa_{\mathrm{xy}} = -2\, w_{,xy} \;.
\end{aligned}$$

For panels with initial geometric imperfections in terms of a stress-free out-of-plane deformation field $\bar{w}$ the elastic in-plane strain-displacement relations are:

$$\begin{aligned}
\epsilon_{\mathrm{x}} &= u_{,x} - w/R_{\mathrm{x}} + w_{,x}(w_{,x}/2 + \bar{w}_{,x}) \;; \\
\epsilon_{\mathrm{y}} &= v_{,y} - w/R_{\mathrm{y}} + w_{,y}(w_{,y}/2 + \bar{w}_{,y}) \;; \\
\gamma_{\mathrm{xy}} &= u_{,y} + v_{,x} - 2\, w/R_{\mathrm{xy}} + w_{,x}(w_{,y} + \bar{w}_{,y}) + w_{,y}\, \bar{w}_{,x} \;.
\end{aligned}$$

These expressions for the strain of weak curved panels apply to the 'compatibility relation':

$$\begin{aligned}
&\epsilon_{\mathrm{x},yy} + \epsilon_{\mathrm{y},xx} - \gamma_{\mathrm{xy},xy} + w_{,yy}/R_{\mathrm{x}} + w_{,xx}/R_{\mathrm{y}} - 2\, w_{,xy}/R_{\mathrm{xy}} \\
&= -w_{,yy}\,(w_{,xx}/2 + \bar{w}_{,xx}) + w_{,xy}\,(w_{,xy} + 2\,\bar{w}_{,xy}) - w_{,xx}\,(w_{,yy}/2 + \bar{w}_{,yy}) \;.
\end{aligned}$$

The strain variations ϵ_{x}, ϵ_{y}, and γ_{xy} can be expressed in the (variations of the) in-plane loads and the curvature terms:

$$\begin{pmatrix} \epsilon_{\mathrm{x}} \\ \epsilon_{\mathrm{y}} \\ \gamma_{\mathrm{xy}} \end{pmatrix} = C \begin{pmatrix} N_{\mathrm{x}} \\ N_{\mathrm{y}} \\ N_{\mathrm{xy}} \end{pmatrix} - K \begin{pmatrix} -w_{,xx} + \gamma_{\mathrm{xz},x} \\ -w_{,yy} + \gamma_{\mathrm{yz},y} \\ -2w_{,xy} + \gamma_{\mathrm{xz},y} + \gamma_{\mathrm{yz},x} \end{pmatrix} \;.$$

The (variations of the) moments in the panel can be expressed with the stiffness relations as:

$$\begin{pmatrix} M_{\mathrm{x}} \\ M_{\mathrm{y}} \\ M_{\mathrm{xy}} \end{pmatrix} = K^{\mathrm{T}} \begin{pmatrix} N_{\mathrm{x}} \\ N_{\mathrm{y}} \\ N_{\mathrm{xy}} \end{pmatrix} + \tilde{D} \begin{pmatrix} -w_{,xx} + \gamma_{\mathrm{xz},x} \\ -w_{,yy} + \gamma_{\mathrm{yz},y} \\ -2w_{,xy} + \gamma_{\mathrm{xz},y} + \gamma_{\mathrm{yz},x} \end{pmatrix} \;.$$

The matrices used here are: $C = A^{-1}$, $K = A^{-1} \cdot B$, and $\tilde{D} = D - B \cdot A^{-1} \cdot B$. Using these expressions for the strains gives for the compatibility relation (without the out-of-plane deformation γ_{xz} and γ_{yz}):

$$\begin{aligned}
&\left(\frac{\partial^2 \cdot}{\partial y^2},\; \frac{\partial^2 \cdot}{\partial x^2},\; -\frac{\partial^2 \cdot}{\partial x\, \partial y} \right) \cdot \left(K \cdot \begin{pmatrix} w_{,xx} \\ w_{,yy} \\ 2\, w_{,xy} \end{pmatrix} + C \cdot \begin{pmatrix} N_{\mathrm{x}} \\ N_{\mathrm{y}} \\ N_{\mathrm{xy}} \end{pmatrix} \right) \\
&\quad + w_{,yy}/R_{\mathrm{x}} - 2\, w_{,xy}/R_{\mathrm{xy}} + w_{,xx}/R_{\mathrm{y}} \\
&= -w_{,yy}\,(w_{,xx}/2 + \bar{w}_{,xx}) + w_{,xy}\,(w_{,xy} + 2\,\bar{w}_{,xy}) - w_{,xx}\,(w_{,yy}/2 + \bar{w}_{,yy}) \;. \qquad (42)
\end{aligned}$$

The non-linear out-of-plane equilibrium equation (similar as (36)) for a 'quasi-shallow' imperfect shell becomes (also without $\gamma_{\rm xz}$ and $\gamma_{\rm yz}$):

$$\left(\frac{\partial^2\cdot}{\partial x^2}\,,\;\frac{\partial^2\cdot}{\partial y^2}\,,\;2\,\frac{\partial^2\cdot}{\partial x\,\partial y}\right)\cdot\left(\tilde{D}\cdot\begin{pmatrix} w_{,xx}\\ w_{,yy}\\ 2\,w_{,xy}\end{pmatrix}-K^{\rm T}\cdot\begin{pmatrix} N_{\rm x}\\ N_{\rm y}\\ N_{\rm xy}\end{pmatrix}\right)$$

$$+\,k_{\rm supp}\,w\;-N_{\rm x}/R_{\rm x}-2\,N_{\rm xy}/R_{\rm xy}-N_{\rm y}/R_{\rm y}$$

$$=N_{\rm x}\,(w_{,xx}+\bar{w}_{,xx})+2\,N_{\rm xy}\,(w_{,xy}+\bar{w}_{,xy})+N_{\rm y}\,(w_{,yy}+\bar{w}_{,yy})\;. \tag{43}$$

With the 'quasi shallow shell assumptions' the terms with the transverse curvature $1/R_{\rm y}$ in the first two stability equations (34) and (35) can be omitted, by which they get the form:
$\partial N_{\rm x}/\partial x+\partial N_{\rm xy}/\partial y=0$; $\partial N_{\rm xy}/\partial x+\partial N_{\rm y}/\partial y=0$.
The in-plane loads apply to the Airy stress function f as follows (see also [25], p.347):
$N_{\rm x}\;=\;f_{,yy}$; $N_{\rm xy}=-f_{,xy}$; $N_{\rm y}\;=\;f_{,xx}$.
Substitution of this stress function in the compatibility equation and expanding the matrix-vector products for anisotropic panels gives:

$$C_{22}\,f_{,xxxx}-2\,C_{26}\,f_{,xxxy}+(2\,C_{12}{+}C_{66})\,f_{,xxyy}-2\,C_{16}\,f_{,xyyy}+C_{11}\,f_{,yyyy}$$

$$+\,K_{21}\,w_{,xxxx}+(2\,K_{26}{-}K_{61})\,w_{,xxxy}+(K_{11}{+}K_{22}{-}2\,K_{66})\,w_{,xxyy}$$
$$+\,(2\,K_{16}{-}K_{62})\,w_{,xyyy}+K_{12}\,w_{,yyyy}+w_{,xx}/R_{\rm y}-2\,w_{,xy}/R_{\rm xy}+w_{,yy}/R_{\rm x}$$

$$=-\,w_{,yy}\,(w_{,xx}/2+\bar{w}_{,xx})+w_{,xy}\,(w_{,xy}+2\,\bar{w}_{,xy})-w_{,xx}\,(w_{,yy}/2+\bar{w}_{,yy})\;.$$

Performing the same for the equilibrium equation gives:

$$\tilde{D}_{11}\,w_{,xxxx}+4\,\tilde{D}_{16}\,w_{,xxxy}+2\,(\tilde{D}_{12}{+}2\,\tilde{D}_{66})\,w_{,xxyy}+4\,\tilde{D}_{26}\,w_{,xyyy}+\tilde{D}_{22}\,w_{,yyyy}$$

$$+\,k_{\rm supp}\,w-K_{21}\,f_{,xxxx}-(2\,K_{26}{-}K_{61})\,f_{,xxxy}-(K_{11}{+}K_{22}{-}2\,K_{66})\,f_{,xxyy}$$
$$-\,(2\,K_{16}{-}K_{62})\,f_{,xyyy}-K_{12}\,f_{,yyyy}-f_{,xx}/R_{\rm y}+2\,f_{,xy}/R_{\rm xy}-f_{,yy}/R_{\rm x}$$

$$=\;f_{,yy}(w_{,xx}+\bar{w}_{,xx})-2\,f_{,xy}(w_{,xy}+\bar{w}_{,xy})+f_{,xx}(w_{,yy}+\bar{w}_{,yy})\;.$$

For the special case of orthotropic panels without geometric twist ($1/R_{\rm xy}=0$) and without shear loading the non-linear compatibility equation for imperfect shells has the orthogonal form

$$C_{22}\,f_{,xxxx}+(2\,C_{12}{+}C_{66})\,f_{,xxyy}+C_{11}\,f_{,yyyy}+K_{21}\,w_{,xxxx}$$

$$+\,(K_{11}{+}K_{22}{-}2\,K_{66})\,w_{,xxyy}+K_{12}\,w_{,yyyy}+w_{,xx}/R_{\rm y}+w_{,yy}/R_{\rm x}$$

$$=-\,w_{,yy}(w_{,xx}/2+\bar{w}_{,xx})+w_{,xy}(w_{,xy}+2\,\bar{w}_{,xy})-w_{,xx}(w_{,yy}/2+\bar{w}_{,yy})\;; \tag{44}$$

and the orthogonal form of the non-linear equilibrium equation

$$\tilde{D}_{11}\,w_{,xxxx}+2\,(\tilde{D}_{12}{+}2\,\tilde{D}_{66}\,)\,w_{,xxyy}+\tilde{D}_{22}\,w_{,yyyy}+k_{\rm supp}\,w$$

$$-\,K_{21}\,f_{,xxxx}-(K_{11}{+}K_{22}{-}2\,K_{66})\,f_{,xxyy}-K_{12}\,f_{,yyyy}-f_{,xx}/R_{\rm y}-f_{,yy}/R_{\rm x}$$

$$=f_{,yy}(w_{,xx}+\bar{w}_{,xx})-2\,f_{,xy}(w_{,xy}+\bar{w}_{,xy})+f_{,xx}(w_{,yy}+\bar{w}_{,yy})\;. \tag{45}$$

B ANALYTICAL BUCKLING SOLUTIONS

This appendix contains a description of some buckling prediction methods based on analytical solutions for panels with uniform curvature, material, and loading. These solutions form the basis of some of the improvements of the 'Design Rules' and can also be used in more simple versions of the panel-based buckling routines. Most of these conventional prediction methods assume untwisted symmetric panels for which $1/R_{\rm xy}$ = 0 and that have a zero B matrix. Asymmetry of the panel-layup in thickness direction can be included by using a so-called 'reduced stiffness matrix' in which the influence of the B matrix is included.

B.1 Solution method for uniform curved orthotropic panels

For most practical solutions the expressions for curved uniform panels are derived analytically using Donnell's strain displacement relations. As shown in section A.6 the in-plane panel loads can be expressed in the Airy stress function f (here for variations that describe the collapse mode):

$N_{\rm x} = f_{,yy}$; $N_{\rm y} = f_{,xx}$; $N_{\rm xy} = -f_{,xy}$.

Using this stress function f in the stability equations (36), (37), and (38) in the linear version of the compatibility equation (42), and in the expression for the moments (43) and reducing these equations for 'quasi-shallow shells' with uniform loading ($N_{\rm x\,0},\ N_{\rm y\,0},\ N_{\rm xy\,0}$) gives:

$$N_{\rm x\,0}\, w_{,xx} + N_{\rm y\,0}\, w_{,yy} + 2\, N_{\rm xy\,0}\, w_{,xy} + f_{,yy}/R_{\rm x} + f_{,xx}/R_{\rm y} - 2\, f_{,xy}/R_{\rm xy}$$
$$+ M_{{\rm x},xx} + M_{{\rm y},yy} + 2\, M_{{\rm xy},xy} - k_{\rm supp}\, w = 0\ .$$

$$A_{55}\, \gamma_{\rm xz} - M_{{\rm x},x} - M_{{\rm xy},y} = 0\ .$$

$$A_{44}\, \gamma_{\rm yz} - M_{{\rm xy},x} - M_{{\rm y},y} = 0\ .$$

$$\left(\frac{\partial^2 .}{(\partial y)^2}\ \ \frac{\partial^2 .}{(\partial x)^2}\ \ \frac{-\partial^2 .}{\partial x\, \partial y} \right) \cdot \left[C \begin{pmatrix} f_{,yy} \\ f_{,xx} \\ -f_{,xy} \end{pmatrix} - K \begin{pmatrix} -w_{,xx} + \gamma_{{\rm xz},x} \\ -w_{,yy} + \gamma_{{\rm yz},y} \\ -2w_{,xy} + \gamma_{{\rm xz},y} + \gamma_{{\rm yz},x} \end{pmatrix} \right]$$
$$+ \frac{w_{,xx}}{R_{\rm y}} + \frac{w_{,yy}}{R_{\rm x}} - 2\frac{w_{,xy}}{R_{\rm xy}} = 0\ .$$

$$\begin{pmatrix} M_{\rm x} \\ M_{\rm y} \\ M_{\rm xy} \end{pmatrix} = K^{\rm T} \begin{pmatrix} f_{,yy} \\ f_{,xx} \\ -f_{,xy} \end{pmatrix} + \tilde{D} \begin{pmatrix} -w_{,xx} + \gamma_{{\rm xz},x} \\ -w_{,yy} + \gamma_{{\rm yz},y} \\ -2w_{,xy} + \gamma_{{\rm xz},y} + \gamma_{{\rm yz},x} \end{pmatrix}\ .$$

Substitution of the expression for the bending moments and expanding the matrix-vector multiplications for orthotropic panels (stiffness matrix elements with indices '16' and '26' are zero) gives for this stability equation, for expressions (37) and (38) for the out-of-plane shear deformation, and for the compatibility equation:

$$-N_{\rm x\,0}\, w_{,xx} - N_{\rm y\,0}\, w_{,yy} - 2\, N_{\rm xy\,0}\, w_{,xy}$$
$$+\tilde{D}_{11}\, w_{,xxxx} + 2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66})\, w_{,xxyy} + \tilde{D}_{22}\, w_{,yyyy} + k_{\rm supp}\, w$$
$$-K_{21}\, f_{,xxxx} - (K_{11} + K_{22} - 2\, K_{66}) f_{,xxyy} - K_{12}\, f_{,yyyy} - f_{,xx}/R_{\rm y} - f_{,yy}/R_{\rm x} + 2\, f_{,xy}/R_{\rm xy}$$
$$- \tilde{D}_{11}\, \gamma_{{\rm xz},xxx} - (\tilde{D}_{12} + 2\,\tilde{D}_{66})\, \gamma_{{\rm xz},xyy} - (\tilde{D}_{12} + 2\,\tilde{D}_{66})\, \gamma_{{\rm yz},xxy} - \tilde{D}_{22}\, \gamma_{{\rm yz},yyy} = 0\ . \quad (46)$$

$$\tilde{D}_{11}\, w_{,xxx} + (\tilde{D}_{12} + 2\,\tilde{D}_{66})\, w_{,xyy} - K_{21}\, f_{,xxx} - (K_{11} - K_{66})\, f_{,xyy}$$
$$+ A_{55}\, \gamma_{\rm xz} - \tilde{D}_{11}\, \gamma_{{\rm xz},xx} - \tilde{D}_{66}\, \gamma_{{\rm xz},yy} - (\tilde{D}_{12} + \tilde{D}_{66})\, \gamma_{{\rm yz},xy} = 0\ . \quad (47)$$

$$(\tilde{D}_{12} + 2\,\tilde{D}_{66})\,w_{,xxy} + \tilde{D}_{22}\,w_{,yyy} - (K_{22} - K_{66})\,f_{,xxy} - K_{12}\,f_{,yyy}$$
$$-\ (\tilde{D}_{12} + \tilde{D}_{66})\gamma_{\mathrm{xz},xy} + A_{44}\,\gamma_{\mathrm{yz}} - \tilde{D}_{66}\,\gamma_{\mathrm{yz},xx} - \tilde{D}_{22}\,\gamma_{\mathrm{yz},yy} = 0\ . \tag{48}$$

$$K_{21}\,w_{,xxxx} + (K_{11} + K_{22} - 2\,K_{66})\,w_{,xxyy} + K_{12}\,w_{,yyyy} + w_{,xx}/R_{\mathrm{y}} + w_{,yy}/R_{\mathrm{x}}$$
$$-2\,w_{,xy}/R_{\mathrm{xy}}\ + C_{22}\,f_{,xxxx} + (2\,C_{12} + C_{66})\,f_{,xxyy} + C_{11}\,f_{,yyyy}$$
$$-\,K_{21}\,\gamma_{\mathrm{xz},xxx} - (K_{11} - K_{66})\,\gamma_{\mathrm{xz},xyy} - (K_{22} - K_{66})\,\gamma_{\mathrm{yz},xxy} - K_{12}\,\gamma_{\mathrm{yz},yyy} = 0\ . \tag{49}$$

These are 4 linear equations in the 4 variables w, f, γ_{xz}, and γ_{yz}. For untwisted panels without shear loading ($1/R_{\mathrm{xy}} = 0,\ N_{\mathrm{xy}\,0} = 0$) these equations apply to the sinusoidal solutions:

$w\ \ = C_{\mathrm{w}}\ \sin(\mu_{\mathrm{x}}\,x)\ \sin(\mu_{\mathrm{y}}\,y)$;
$f\ \ = C_{\mathrm{f}}\ \sin(\mu_{\mathrm{x}}\,x)\ \sin(\mu_{\mathrm{y}}\,y)$;
$\gamma_{\mathrm{xz}} = C_{\mathrm{xz}}\ \cos(\mu_{\mathrm{x}}\,x)\ \sin(\mu_{\mathrm{y}}\,y)$;
$\gamma_{\mathrm{yz}} = C_{\mathrm{yz}}\ \sin(\mu_{\mathrm{x}}\,x)\ \cos(\mu_{\mathrm{y}}\,y)$.

Here $\mu_{\mathrm{x}} = \pi/L$ and $\mu_{\mathrm{y}} = \pi/b$ with L the half-wave length and b the half-wave width of the deformation pattern. Substitution of these functions in the differential equations ((46), (47), (48), and (49)) for a non-trivial solution gives the set of characteristic equations:

$$[\,\mu_{\mathrm{x}}^2\,N_{\mathrm{x}\,0} + \mu_{\mathrm{y}}^2\,N_{\mathrm{y}\,0} + \mu_{\mathrm{x}}^4\,\tilde{D}_{11} + 2\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_{\mathrm{y}}^4\,\tilde{D}_{22} + k_{\mathrm{supp}}\,]\,C_{\mathrm{w}}$$
$$-[\,\mu_{\mathrm{x}}^4\,K_{21} + \mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}^2\,(K_{11} + K_{22} - 2\,K_{66}) + \mu_{\mathrm{y}}^4\,K_{12} - \mu_{\mathrm{x}}^2/R_{\mathrm{y}} - \mu_{\mathrm{y}}^2/R_{\mathrm{x}}\,]\,C_{\mathrm{f}}$$
$$-[\,\mu_{\mathrm{x}}^3\,\tilde{D}_{11}\ + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66})\,]\,C_{\mathrm{xz}} - [\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_{\mathrm{y}}^3\,\tilde{D}_{22}\,]\,C_{\mathrm{yz}} = 0\ ;$$

$$[\,\mu_{\mathrm{x}}^4\,K_{21} + \mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}^2\,(K_{11} + K_{22} - 2\,K_{66}) + \mu_{\mathrm{y}}^4\,K_{12} - \mu_{\mathrm{x}}^2/R_{\mathrm{y}} - \mu_{\mathrm{y}}^2/R_{\mathrm{x}}\,]\,C_{\mathrm{w}}$$
$$+[\,\mu_{\mathrm{x}}^4\,C_{22} + \mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}^2\,(2\,C_{12} + C_{66}) + \mu_{\mathrm{y}}^4\,C_{11}\,]\,C_{\mathrm{f}}$$
$$-[\,\mu_{\mathrm{x}}^3\,K_{21} + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}^2\,(K_{11} - K_{66})\,]\,C_{\mathrm{xz}} - [\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}\,(K_{22} - K_{66}) + \mu_{\mathrm{y}}^3\,K_{12}\,]\,C_{\mathrm{yz}} = 0\ ;$$

$$-[\,\mu_{\mathrm{x}}^3\,\tilde{D}_{11} + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66})\,]\,C_{\mathrm{w}} + [\,\mu_{\mathrm{x}}^3\,K_{21} + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}^2\,(K_{11} - K_{66})\,]\,C_{\mathrm{f}}$$
$$+[\,A_{55} + \mu_{\mathrm{x}}^2\,\tilde{D}_{11} + \mu_{\mathrm{y}}^2\,\tilde{D}_{66}\,]\,C_{\mathrm{xz}} + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}\,(\tilde{D}_{12} + \tilde{D}_{66})\,C_{\mathrm{yz}}\ = 0\ ;$$

$$-[\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_{\mathrm{y}}^3\,\tilde{D}_{22}\,]\,C_{\mathrm{w}} + [\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}\,(K_{22} - K_{66}) + \mu_{\mathrm{y}}^3\,K_{12}\,]\,C_{\mathrm{f}}$$
$$+\mu_{\mathrm{x}}\,\mu_{\mathrm{y}}\,(\tilde{D}_{12} + \tilde{D}_{66})\,C_{\mathrm{xz}} + [\,A_{44} + \mu_{\mathrm{x}}^2\,\tilde{D}_{66} + \mu_{\mathrm{y}}^2\,\tilde{D}_{22}\,]\,C_{\mathrm{yz}}\ = 0\ .$$

This set of equations can be written in matrix-form:

$$\begin{pmatrix} \mu_{\mathrm{x}}^2\,N_{\mathrm{x}\,0} + \mu_{\mathrm{y}}^2\,N_{\mathrm{y}\,0} + H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12} & H_{22} & H_{23} & H_{24} \\ H_{13} & H_{23} & H_{33} & H_{34} \\ H_{14} & H_{24} & H_{34} & H_{44} \end{pmatrix} \cdot \begin{pmatrix} C_{\mathrm{w}} \\ -C_{\mathrm{f}} \\ C_{\mathrm{xz}} \\ C_{\mathrm{yz}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\ .$$

The applied loads $N_{\mathrm{x}\,0}$ and $N_{\mathrm{y}\,0}$ can be scaled with λ such that this set of equations is satisfied.

A numerical solution expressed in (sub)determinants has the risk of overflow, especially for non-sandwich panels or sandwich panels with a very stiff core. This overflow can be avoided by multiplying the 3-rd row and 3-rd column with the square-root of the longitudinal out-of-plane shear flexibility A_{55}^{-1} and multiplying the 4-th row and 4-th column with the square-root of the transverse out-of-plane shear flexibility A_{44}^{-1}. This gives for the matrix elements of the 'reduced' characteristic equation:

$$H_{11} = \mu_x^4\,\tilde{D}_{11} + 2\,\mu_x^2\,\mu_y^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_y^4\,\tilde{D}_{22} + k_{\mathrm{supp}}\ ;$$

$$H_{22} = -\,[\,\mu_x^4\,C_{22} + \mu_x^2\,\mu_y^2\,(2\,C_{12} + C_{66}) + \mu_y^4\,C_{11}\,]\ ;$$

$$H_{12} = \mu_x^4\,K_{21} + \mu_x^2\,\mu_y^2\,(K_{11} + K_{22} - 2\,K_{66}) + \mu_y^4\,K_{12} - \mu_x^2/R_y - \mu_y^2/R_x\ ;$$

$$H_{33}^* = 1 + A_{55}^{-1}\,(\mu_x^2\,\tilde{D}_{11} + \mu_y^2\,\tilde{D}_{66})\ ;$$

$$H_{13}^* = -\,[\,\mu_x^3\,\tilde{D}_{11} + \mu_x\,\mu_y^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66})\,]\,\sqrt{A_{55}^{-1}}\ ;$$

$$H_{23}^* = -\,[\,\mu_x^3\,K_{21} + \mu_x\,\mu_y^2\,(K_{11} - K_{66})\,]\,\sqrt{A_{55}^{-1}}\ ;$$

$$H_{44}^* = 1 + A_{44}^{-1}\,(\mu_x^2\,\tilde{D}_{66} + \mu_y^2\,\tilde{D}_{22})\ ;$$

$$H_{14}^* = -\,[\,\mu_x^2\,\mu_y\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_y^3\,\tilde{D}_{22}\,]\,\sqrt{A_{44}^{-1}}\ ;$$

$$H_{24}^* = -\,[\,\mu_x^2\,\mu_y\,(K_{22} - K_{66}) + \mu_y^3\,K_{12}\,]\,\sqrt{A_{44}^{-1}}\ ;$$

$$H_{34}^* = \mu_x\,\mu_y\,(\tilde{D}_{12} + \tilde{D}_{66})\,\sqrt{A_{44}^{-1}\,A_{55}^{-1}}\ .$$

Using the load factor λ the matrix-form of the characteristic equation is:

$$\det\begin{pmatrix} \lambda(\mu_x^2\,N_{x0} + \mu_y^2\,N_{y0}) + H_{11} & H_{12} & H_{13}^* & H_{14}^* \\ H_{12} & H_{22} & H_{23}^* & H_{24}^* \\ H_{13}^* & H_{23}^* & H_{33}^* & H_{34}^* \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44}^* \end{pmatrix} = 0 \tag{50}$$

The general solution for λ is thus: $\lambda\,(-\,\mu_x^2\,N_{x0} - \mu_y^2\,N_{y0}) = \det(H)/\mathrm{subdet}(H_{11})$.

Here $\mathrm{subdet}(H_{11})$ is the determinant of the 3x3 matrix that remains after omission of the first row and first column from matrix H .

For sandwich panels with moderate out-of-plane shear flexibility and with not too strong asymmetry in the layup, the terms H_{23}^* and H_{24}^* are relatively small. Omission of these terms in the determinant equation for the critical load factor gives a rather practical expression:

$$\lambda\,(-\,\mu_x^2\,N_{x0} - \mu_y^2\,N_{y0}) = H_{11} - (H_{12}^*)^2/H_{22} + termA/termB \tag{51}$$

With $termA = 2\,H_{13}^*\,H_{14}^*\,H_{34}^* - H_{33}^*\,(H_{14}^*)^2 - H_{44}^*\,(H_{13}^*)^2$ and $termB = H_{33}^*\,H_{44}^* - (H_{34}^*)^2$.

For symmetric sandwich panels (zero K matrix) this solution is identical to that of C.W. Bert, Crisman, and Nordby [1, 6]. It must be noted that the description of the solution given in [6] is for sandwich panels with relatively thin load-carrying facings and transverse curvature only.

For non-sandwich panels and sandwich panels with very stiff core the terms A_{55}^{-1} and A_{44}^{-1} approach zero, for which the fraction $termA/termB$ approaches zero and the expression for buckling is simply: $\lambda\,(-\,\mu_x^2\,N_{x0} - \mu_y^2\,N_{y0}) = H_{11} + H_{12}^2/(-H_{22})$.
This solution is similar to that given in section 3.1.1 of the GARTEUR report by B. Geier [8], where $-H_{22}$ is named G_{11} , H_{12} is named G_{12} , and H_{11} is named G_{22} .

For panels without longitudinal curvature the contribution of the asymmetry matrix K can be expressed in an 'effective' transverse curvature such that the expression for H_{12} is conserved:

$$1/R_y^* = 1/R_y - \mu_y^2\,\Big((\mu_x/\mu_y)^2\,K_{21} + (\mu_y/\mu_x)^2\,K_{12} + K_{11} + K_{22} - 2\,K_{66}\Big)\ . \tag{52}$$

Here $\mu_x/\mu_y = b/L$ can be evaluated with the solution for flat plates in section B.4.

B.2 Solution for slightly anisotropic non-sandwich panels

Because of the aerodynamic twist of a rotor blade the panels in a blade cross section have some twist, while the edgewise shear loading in a blade cross section results in shear loads in the panels. Both the shear loading and the geometric twist introduce odd derivatives in the set of stability equations, while their contribution may either partially compensate or add-up to each other. For investigations into buckling of blades with anisotropic material properties (for blades with Bending-Torsion coupling) and eventually with the effect of shear loading and geometric twist a relatively simple and approximate solution method is given for non-sandwich panels.

Assuming no out-of-plane shear deformation ($\gamma_{\rm xz} = 0$ and $\gamma_{\rm yz} = 0$) the equations (46) and (49) get the form

$$\begin{aligned} &-N_{\rm x\,0}\, w_{,xx} - N_{\rm y\,0}\, w_{,yy} - 2\, N_{\rm xy\,0}\, w_{,xy} \\ &+\tilde{D}_{11}\, w_{,xxxx} + 2\,(\tilde{D}_{12}{+}2\,\tilde{D}_{66})\, w_{,xxyy} + \tilde{D}_{22}\, w_{,yyyy} + k_{\rm supp}\, w \\ &-K_{21}\, f_{,xxxx} - (K_{11}{+}K_{22}{-}2\,K_{66}) f_{,xxyy} - K_{12}\, f_{,yyyy} \\ &- f_{,xx}/R_{\rm y} - f_{,yy}/R_{\rm x} + 2\, f_{,xy}/R_{\rm xy} = 0\ . \end{aligned} \tag{53}$$

$$\begin{aligned} &K_{21}\, w_{,xxxx} + (K_{11}{+}K_{22}{-}2\,K_{66})\, w_{,xxyy} + K_{12}\, w_{,yyyy} + w_{,xx}/R_{\rm y} + w_{,yy}/R_{\rm x} - 2\, w_{,xy}/R_{\rm xy} \\ &+ C_{22}\, f_{,xxxx} - 2\, C_{26}\, f_{,xxxy} + (2\, C_{12}{+}C_{66})\, f_{,xxyy} - 2\, C_{16}\, f_{,xyyy} + C_{11}\, f_{,yyyy} = 0\ . \end{aligned} \tag{54}$$

This problem has some similarities with that analysed by Tennyson and Muggeridge [24] for anisotropic curved panels, which also includes terms for K_{16}, K_{26}, K_{61}, and K_{62}. Reducing the solution of Tennyson and Muggeridge for slightly anisotropic plates without shear loading gives:

$$\lambda\,(-\,\mu_{\rm x}^2\, N_{\rm x\,0} - \mu_{\rm y}^2\, N_{\rm y\,0}) = H_{11} + H_{12}^{\,2}/(-H_{22} - termC^2/(-H_{22}))\ . \tag{55}$$

Here $termC$ stands for $termC = 2\,\mu_{\rm x}\,\mu_{\rm y}\,(\mu_{\rm x}^2\, C_{26} + \mu_{\rm y}^2\, C_{16})$.

B.3 Routine for buckling of uniform curved sandwich panels

For non-twisted panels with orthotropic material properties and without shear loading, the stability equations and the compatibility relation can be expressed in the deformations w, $\gamma_{\rm xz}$, $\gamma_{\rm yz}$, and the Airy stress function f . These linear equations have either only even derivatives or only odd derivatives in both x and y. For those equations the solution of the collapse mode is sinusoidal in both x and y direction. A derivation of those linear equations is given in section B.1 while these equations are implemented in routine *rlamod3* that returns the critical load factor for a given length of the deformation pattern, see section D.3.

For a value 2 of the input item '*imethd*', routine *rlamod3* returns the factor λ on the applied loads $N_{\rm x\,0}$ and $N_{\rm y\,0}$ that is calculated from:

$$\lambda\,(-\,\mu_{\rm x}^2\, N_{\rm x\,0} - \mu_{\rm y}^2\, N_{\rm y\,0}) = H_{11} + H_{12}^{\,2}/(-H_{22} - termC^2/(-H_{22})) + termA/termB \tag{56}$$

The solution (56) is a combination of (51) and (55) as presented in sections B.1 and B.2

In the calculation of the critical load factor, several (sinusoidal) half-waves in transverse direction are traced: $\sin(\pi\, y/b)$, $\sin(2\,\pi\, y/b)$, $\sin(3\,\pi\, y/b)$, etcetera. In this analytical solution, both the longitudinal and transverse curvature are taken into account.

The routine *rlamod3* that returns the load-factor for buckling is described in section D.3.

The panel-averaged stiffnesses are calculated using the 'weighting functions' in section C.1.

B.4 Linearised solution for axially loaded symmetric sandwich plates

For flat symmetric plates the curvatures $1/R_{\mathrm{x}}$, $1/R_{\mathrm{y}}$, and $1/R_{\mathrm{xy}}$ and the elementgs of the K matrix are zero. This means that buckling is described with (46), (47), and (48) only, so without the compatibility equation(49) for in-plane load variations The latter can also be concluded from symmetry with respect to the plane midsurface .

For the set of characteristic equations the independence from the in-plane compatibility equation means that the terms H_{12}, H_{23}^*, and H_{24}^* are zero. The expression for buckling of flat plates is thus equal to (51) without the H_{12} terms:

$$\lambda\,(-\mu_{\mathrm{x}}^2\,N_{\mathrm{x}\,0} - \mu_{\mathrm{y}}^2\,N_{\mathrm{y}\,0}) = H_{11} + \frac{2\,H_{13}^*\,H_{14}^*\,H_{34}^* - H_{33}^*\,(H_{14}^*)^2 - H_{44}^*\,(H_{13}^*)^2}{H_{44}^*\,H_{33}^* - (H_{34}^*)^2}\ .$$

For very long sandwich plates the collapse mode will have a half-wave length with the smallest load factor λ. Finding an analytical solution for the half-wave length with the smallest λ value is rather complicated because of the strong non-linearity of the rightmost term with the sandwich properties. For a sandwich plate with relatively small out-of-plane shear flexibilities all non-linear terms in these flexibilities A_{44}^{-1} and A_{55}^{-1} can be omitted, which leaves:

$$\lambda\,(-\mu_{\mathrm{x}}^2\,N_{\mathrm{x}\,0} - \mu_{\mathrm{y}}^2\,N_{\mathrm{y}\,0}) = H_{11} - (H_{14}^*)^2 - (H_{13}^*)^2$$

$$= \mu_{\mathrm{x}}^4\,\tilde{D}_{11} + 2\,\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_{\mathrm{y}}^4\,\tilde{D}_{22}$$

$$-\,A_{55}^{-1}\,(\mu_{\mathrm{x}}^3\,\tilde{D}_{11} + \mu_{\mathrm{x}}\,\mu_{\mathrm{y}}^2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}))^2 - A_{44}^{-1}\,(\mu_{\mathrm{x}}^2\,\mu_{\mathrm{y}}\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \mu_{\mathrm{y}}^3\,\tilde{D}_{22})^2\ .$$

To find the smallest λ value express the half-wave length L as $L = \chi\,b$, so that $\mu_{\mathrm{x}} = \mu_{\mathrm{y}}/\chi$. Substitution of this relation in the expression for the critical load factor gives an expression for λ in terms of the variable χ (for axial compression only):

$$\lambda\,\mu_{\mathrm{y}}^2\,(-N_{\mathrm{x}\,0}) = \mu_{\mathrm{y}}^4\,(\tilde{D}_{11}/\chi^2 + 2\,(\tilde{D}_{12} + 2\,\tilde{D}_{66}) + \chi^2\,\tilde{D}_{22})$$

$$-\,\mu_{\mathrm{y}}^6\left(A_{55}^{-1}\,(\tilde{D}_{11}/\chi^2 + \tilde{D}_{12} + 2\,\tilde{D}_{66})^2 + A_{44}^{-1}\,(\tilde{D}_{12} + 2\,\tilde{D}_{66} + \chi^2\,\tilde{D}_{22})^2/\chi^2\right)\ .$$

The smallest ('critical') value for λ follows from a zero value of the derivative $\partial\lambda/\partial(\chi^2) = 0$.

$$0 \;=\; \mu_{\mathrm{y}}^4\,(-\tilde{D}_{11}/\chi^4 + \tilde{D}_{22})$$

$$+\mu_{\mathrm{y}}^6\left(A_{55}^{-1}\,2\,\tilde{D}_{11}\,(\tilde{D}_{11}/\chi^2 + \tilde{D}_{12} + 2\,\tilde{D}_{66})/\chi^4 + A_{44}^{-1}\,(\,(\tilde{D}_{12} + 2\,\tilde{D}_{66})^2/\chi^4 - \tilde{D}_{22}^2)\right)\ .$$

For non-sandwich plates the second term in this relation vanishes so that the smallest factor λ is found for the critical half-wave length $\chi^2 = \sqrt{\tilde{D}_{11}/\tilde{D}_{22}}$ and likewise $L_{\mathrm{crit}} = b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}$. For sandwich plates with small flexibilities A_{55}^{-1} and A_{44}^{-1} the influence of these terms on the collapse mode is also small so that the governing terms in the condition for the critical load factor λ can be evaluated with the χ value for non-sandwich plates:

$$0 \;=\; \mu_{\mathrm{y}}^4\,(-\tilde{D}_{11}/\chi^4 + \tilde{D}_{22}) + \mu_{\mathrm{y}}^6\,A_{55}^{-1}\,2\,\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\;\tilde{D}_{22}\,(1+\beta) + \mu_{\mathrm{y}}^6\,A_{44}^{-1}\,\tilde{D}_{22}^2(\beta^2 - 1)\ .$$

This equation is satisfied for

$$(1/\chi^2) = \sqrt{\tilde{D}_{22}/\tilde{D}_{11}}\,\sqrt{1 + \mu_{\mathrm{y}}^2\,A_{55}^{-1}\,\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\,(2 + 2\,\beta) + \mu_{\mathrm{y}}^2\,A_{44}^{-1}\,\tilde{D}_{22}\,(\beta^2 - 1)}\ .$$

Based on this formulation the following sandwich stiffness ratios are introduced:

$$r_1 = \mu_{\rm y}^2 \, (\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\;/A_{55})\,(1+\beta)/2 \quad \text{and} \quad r_2 = \mu_{\rm y}^2\,(\tilde{D}_{22}/A_{44})\,(1+\beta)/2\;. \tag{57}$$

Using these stiffness ratios r_1 and r_2 gives: $(1/\chi^2) = \sqrt{\tilde{D}_{22}/\tilde{D}_{11}}\,(1 + 2\,r_1 + r_2\,(\beta - 1))$.
The critical half-wave length applies to

$$L_{\rm crit} \; = b\,\sqrt[4]{\tilde{D}_{11}/\tilde{D}_{22}}\,(1 + r_2/2)/(1 + r_1 + \beta\,r_2/2)\;. \tag{58}$$

And finally the expression for the buckling load factor with moderate sandwich properties is:

$$\lambda_{\rm cr}\,(-N_{\rm x\,0}) = \mu_{\rm y}^2\,\sqrt{\tilde{D}_{22}/\tilde{D}_{11}}\,(2 + 2\,\beta)/(1 + r_1 + r_2)\;. \tag{59}$$

These expressions for the critical half-wave length and for the critical load factor are formulated such that they avoid 'division-by-zero' in case of large values of r_1 or r_2 .

B.5 Solution for symmetric sandwich plates including shear loading

Compared to the solution for curved orthotropic panels described in section B.1 the shear-web of a rotor blade is also loaded by shear $N_{\rm xy\,0}$. Using the fact that the shear web of a rotor blade is flat and the (sandwich) layup is usually symmetric allows a relatively simple solution.

After omission of the curvature terms $1/R_{\rm x}$, $1/R_{\rm y}$, and $1/R_{\rm xy}$, after omission of the elements of the K matrix (that describes asymmetric layup) from equations (36), (37), and (38), and after substitution of the bending moments by the orthotropic stiffness relations, the resulting set of equations in w , in $\gamma_{\rm xz}$, and in $\gamma_{\rm yz}$ become:

$$\begin{aligned} &- N_{\rm x\,0}\,w_{,xx} - N_{\rm y\,0}\,w_{,yy} - 2\,N_{\rm xy\,0}\,w_{,xy} \\ &+ \tilde{D}_{11}\,w_{,xxxx} + 2\,(\tilde{D}_{12}{+}2\,\tilde{D}_{66})\,w_{,xxyy} + \tilde{D}_{22}\,w_{,yyyy} + k_{\rm supp}\,w \\ &- \tilde{D}_{11}\,\gamma_{{\rm xz},xxx} - (\tilde{D}_{12}+2\,\tilde{D}_{66})\,\gamma_{{\rm xz},xyy} - (\tilde{D}_{12}+2\,\tilde{D}_{66})\,\gamma_{{\rm yz},xxy} - \tilde{D}_{22}\,\gamma_{{\rm yz},yyy} = 0\;, \end{aligned} \tag{60}$$

$$\begin{aligned} &\tilde{D}_{11}\,w_{,xxx} + (\tilde{D}_{12}+2\,\tilde{D}_{66})\,w_{,xyy} \\ &+ A_{55}\,\gamma_{\rm xz} - \tilde{D}_{11}\,\gamma_{{\rm xz},xx} - \tilde{D}_{66}\,\gamma_{{\rm xz},yy} - (\tilde{D}_{12}+\tilde{D}_{66})\,\gamma_{{\rm yz},xy} = 0\;, \end{aligned} \tag{61}$$

$$\begin{aligned} &(\tilde{D}_{12}+2\,\tilde{D}_{66})\,w_{,xxy} + \tilde{D}_{22}\,w_{,yyy} \\ &-\;(\tilde{D}_{12}+\tilde{D}_{66})\gamma_{{\rm xz},xy} + A_{44}\,\gamma_{\rm yz} - \tilde{D}_{66}\,\gamma_{{\rm yz},xx} - \tilde{D}_{22}\,\gamma_{{\rm yz},yy} = 0\;. \end{aligned} \tag{62}$$

This set of equations is used as basis for a more detailed prediction method of shear web buckling. The shear web of most wind turbine rotor blades is characterised by a sandwich layup with a relatively thick core. For this layup the failure mode is characterised by a short half-wave length, of which the 'wrinkles' make some angle ϕ with the spanwise direction. For this collapse mode the elastic deformation perpendicular to the wrinkles is higher than the elastic deformation for bending in the length of the wrinkles. For this reason it is conservative to describe this wrinkling failure mode of thick sandwich plates as a prismatic deformation pattern.
For wrinkles that make an angle ϕ with the blade axis the collapse mode can be described with:

$$\begin{aligned} w(x,y) \;&=\; C_{\rm w}\,\sin(\,\mu_{\rm b}\,(y\,\cos\phi - x\,\sin\phi)\,)\;, \\ \gamma_{\rm xz}(x,y) &=\; C_{\rm xz}\,\cos(\,\mu_{\rm b}\,(y\,\cos\phi - x\,\sin\phi)\,)\;, \\ \gamma_{\rm yz}(x,y) &= -\,C_{\rm yz}\,\cos(\,\mu_{\rm b}\,(y\,\cos\phi - x\,\sin\phi)\,)\;. \end{aligned}$$

Here $\mu_\mathrm{b} = \pi / b$ where b denotes the half-wave length of the buckles or 'wrinkles'.
Substitution of this collapse mode in the equations (60) to (62) gives:

$$\mu_\mathrm{b}^2 \left(N_{\mathrm{x}\,0} \, (\sin\phi)^2 + N_{\mathrm{y}\,0} \, (\cos\phi)^2 - 2 \, N_{\mathrm{xy}\,0} \, \sin\phi \, \cos\phi \right) C_\mathrm{w}$$

$$+ \mu_\mathrm{b}^4 \left(\tilde{D}_{11} \, (\sin\phi)^4 + 2 \, (\tilde{D}_{12} + 2 \, \tilde{D}_{66}) \, (\sin\phi)^2 \, (\cos\phi)^2 + \tilde{D}_{22} \, (\cos\phi)^4 \right) C_\mathrm{w} + k_\mathrm{supp} \, C_\mathrm{w}$$

$$+ \mu_\mathrm{b}^3 \left(\tilde{D}_{11} \, (\sin\phi)^3 + (\tilde{D}_{12} + 2 \, \tilde{D}_{66}) \, \sin\phi \, (\cos\phi)^2 \right) C_\mathrm{xz}$$

$$+ \mu_\mathrm{b}^3 \left((\tilde{D}_{12} + 2 \, \tilde{D}_{66}) \, (\sin\phi)^2 \, \cos\phi + \tilde{D}_{22} \, (\cos\phi)^3 \right) C_\mathrm{yz} = 0 \; ,$$

$$\mu_\mathrm{b}^3 \left(\tilde{D}_{11} \, (\sin\phi)^3 + (\tilde{D}_{12} + 2 \, \tilde{D}_{66}) \, \sin\phi \, (\cos\phi)^2 \right) C_\mathrm{w}$$

$$+ \left(A_{55} + \mu_\mathrm{b}^2 \, (\tilde{D}_{11} \, (\sin\phi)^2 + \tilde{D}_{66} \, (\cos\phi)^2 \right) C_\mathrm{xz}$$

$$+ \mu_\mathrm{b}^2 \, (\tilde{D}_{12} + \tilde{D}_{66}) \, \sin\phi \, \cos\phi \; C_\mathrm{yz} = 0 \; ,$$

$$\mu_\mathrm{b}^3 \left((\tilde{D}_{12} + 2 \, \tilde{D}_{66}) \, (\sin\phi)^2 \, \cos\phi + \tilde{D}_{22} \, (\cos\phi)^3 \right) C_\mathrm{w}$$

$$+ \mu_\mathrm{b}^2 \, (\tilde{D}_{12} + \tilde{D}_{66}) \cos\phi \, \sin\phi \; C_\mathrm{xz}$$

$$+ \left(A_{44} + \mu_\mathrm{b}^2 \, (\, \tilde{D}_{66} \, (\sin\phi)^2 + \tilde{D}_{22} \, (\cos\phi)^2 \,) \right) C_\mathrm{yz} = 0 \; .$$

This set of characteristic equations can be written in matrix-form:

$$\begin{pmatrix} \mu_\mathrm{b}^2 \begin{pmatrix} N_{\mathrm{x}\,0} \, (\sin\phi)^2 + N_{\mathrm{y}\,0} \, (\cos\phi)^2 \\ - 2 \, N_{\mathrm{xy}\,0} \, \sin\phi \, \cos\phi \end{pmatrix} + H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{pmatrix} \cdot \begin{pmatrix} C_\mathrm{w} \\ C_\mathrm{xz} \\ C_\mathrm{yz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; .$$

A trivial solution is given by zero values of C_w , C_xz , and C_yz .
A non-trivial solution is obtained if the determinant of the matrix equals zero, which is the case for some factor λ on the applied loading:

$$\lambda \, \mu_\mathrm{b}^2 \, (2 \, N_{\mathrm{xy}\,0} \, \sin\phi \, \cos\phi - N_{\mathrm{x}\,0} \, (\sin\phi)^2 - N_{\mathrm{y}\,0} \, (\cos\phi)^2) = \det(H) / (H_{22} \, H_{33} - H_{23}^2) \; . \quad (63)$$

In this solution the matrix elements $H_{i\,j}$ depend on the orientation ϕ and the half-wave length of the wrinkles. The most critical collapse mode can be found by (numerical) searching along the orientation ϕ and parameter μ_b for the smallest load factor λ_cr . With regard to the finite width of the shear web and the assumption that the wrinkles are prismatic, it is not realistic to consider wrinkles with a half-wave length that is larger than the shear web width in this search process.

In a numerical solution of (63) the r.h.s. term may have over-flow for large values of A_{44} and A_{55} which is the case for non-sandwich panels or sandwich panels with a very stiff core. Similar as for the solution described in section B.1 this overflow can be avoided by dividing the 2-nd row and 2-nd column by $\sqrt(A_{55})$ and by dividing the 3-rd row and 3-rd column by $\sqrt(A_{44})$.

For a sandwich shear web that is (also) loaded by in-plane bending, the axial in-plane loading varies over the web width. It is un-conservative to use the 'web-width-average' axial loading while it is somewhat conservative to use the largest compressive (most negative) loading in the panel edge. A fairly reasonable compromise is to use the compressive loading in the shear web at $b/3$ from the 'most compressive' loaded edge.

B.6 Face wrinkling failure modes of sandwich panels

The solution described in the previous section applies to panels with moderate to small out-of-plane shear flexibility, of which the collapse mode can be described by panel bending.
Sandwich panels with a soft core and facings that are thin compared with the core thickness may show 'face wrinkling' when loaded by compression. For this face wrinkling failure mode deformation energy is absorbed by bending of the facings, by shear deformation of the core, and also by compression of the core in thickness-direction. For this 'face wrinkling' failure mode an analytical solution was derived in the BUCKBLADE project (section 4.2.4 on pp.30-32 of [13]).

B.6.1 General solution for wrinkling of curved panels

A relatively thick core serves as an elastic support for 'wrinkling' of the facings. For geometrical perfect curved orthotropic panels with an elastic support and with uniform loading and -deformation pattern in transverse direction the stability is described by the single equation:

$$(\tilde{D}_{11\,\mathrm{f}} + \frac{K_{21\,\mathrm{f}}^{\;2}}{C_{22\,\mathrm{f}}})\, w_{,xxxx} + 2\, \frac{K_{21\,\mathrm{f}}}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}}\, w_{,xx} + (k_{\mathrm{supp}} + \frac{1}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}^2})\, w = N_{\mathrm{x\,f}}\, w_{,xx}\;. \qquad (64)$$

Here $\tilde{D}_{11\,\mathrm{f}},\; K_{21\,\mathrm{f}},$ and $C_{22\,\mathrm{f}}$ are stiffnesses of one facing and $N_{\mathrm{x\,f}}$ is the axial load in one facing. If the stiffness terms are for a complete panel, then the solution is that for a box-structure with an elatic interior foam. Depending on the expression of the support stiffness k_{supp} this equation may apply to face wrinkling of symmetric orthotropic sandwich panels. A solution of this equation is of the form $w(x) = w\,\sin(\mu_{\mathrm{x}}\, x)$.

For a core of a symmetric sandwich panel face wrinkling is the most critical if the collapse mode is anti-symmetric with respect to the core mid-surface. For this mode the support stiffness is $k_{\mathrm{supp}} = 2\, G_{\mathrm{core}}\, \mu_{\mathrm{x}}\, \sinh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}/2)/\cosh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}/2)$. The stability equation is then

$$-\lambda\, \mu_{\mathrm{x}}^2\, N_{\mathrm{x\,f}} = (\tilde{D}_{11\,\mathrm{f}} + \frac{K_{21\,\mathrm{f}}^{\;2}}{C_{22\,\mathrm{f}}})\, \mu_{\mathrm{x}}^4 - \frac{2\, K_{21\,\mathrm{f}}}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}}\, \mu_{\mathrm{x}}^2 + 2\, G_{\mathrm{core}}\, \mu_{\mathrm{x}} \frac{\sinh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}/2)}{\cosh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}/2)} + \frac{1}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}^2}\;.$$

For the most critical collapse mode one has to find the minimum ('critical') value of the factor λ_{cr} on the axial loading $N_{\mathrm{x\,f}}$. This minimum is found for a wave-length parameter μ_{x} that is the positive solution of

$$(\tilde{D}_{11\,\mathrm{f}} + \frac{K_{21\,\mathrm{f}}^{\;2}}{C_{22\,\mathrm{f}}})\, \mu_{\mathrm{x}}^4 - G_{\mathrm{core}}\, \mu_{\mathrm{x}}\, \frac{\sinh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}) - \mu_{\mathrm{x}}\, t_{\mathrm{c}}}{\cosh(\mu_{\mathrm{x}}\, t_{\mathrm{c}}) + 1} - \frac{1}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}^2} = 0\;.$$

An analytical solution for this equation is hard to find.

Since face-wrinkling occurs for panels with a flexible core ($G_{\mathrm{core}}\, t_{\mathrm{c}}^3 < \tilde{D}_{11\,\mathrm{f}}$) and because the equations are derived for a longitudinal half-wave length that is small, a solution for face wrinkling will be sought by linearisation for large values of $(\mu_{\mathrm{x}}\, t_{\mathrm{c}})$ instead. This means that the support stiffness can be approximated with $k_{\mathrm{supp}} = 2\, G_{\mathrm{core}}\, \mu_{\mathrm{x}}$ and the characteristic equation gets the form

$$-\,\lambda\, \mu_{\mathrm{x}}^2\, N_{\mathrm{x\,f}} = (\tilde{D}_{11\,\mathrm{f}} + \frac{K_{21\,\mathrm{f}}^{\;2}}{C_{22\,\mathrm{f}}})\, \mu_{\mathrm{x}}^4 - 2\, \frac{K_{21\,\mathrm{f}}}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}}\, \mu_{\mathrm{x}}^2 + 2\, G_{\mathrm{core}}\, \mu_{\mathrm{x}} + \frac{1}{C_{22\,\mathrm{f}}\, R_{\mathrm{y}}^2}\;. \qquad (65)$$

The most buckling-critical value λ_{cr} has a minimum value for a wave-length parameter that is the (only!) positive solution of

$$(\tilde{D}_{11} + K_{21}^2/C_{22})\, \mu_{\mathrm{x}}^4 - G_{\mathrm{core}}\, \mu_{\mathrm{x}} - 1/(C_{22}\, R_{\mathrm{y}}^2) = 0\;.$$

B.6.2 Solution for wrinkling of flat plates

For flat plates the value of μ_{x} that gives the smallest axial loading follows from solving equation (65) for $1/R_{\mathrm{y}} = 0$, which gives: $\mu_{\mathrm{x}} = \sqrt[3]{G_{\mathrm{core}}/(\tilde{D}_{11\,\mathrm{f}} + K_{21\,\mathrm{f}}^{\;2}/C_{22\,\mathrm{f}})}$.

Assuming that only the facings carry panel loading, the corresponding axial critical compressive panel-load for face wrinkling follows after substitution in (65):

$$-N_{\mathrm{x\,cr}} = \sum_{\mathrm{facings}} -\lambda\, N_{\mathrm{x\,f}} = \sum_{\mathrm{facings}} 3\,\sqrt[3]{G_{\mathrm{core}}^{\;2} \cdot (\tilde{D}_{11\,\mathrm{f}} + K_{21\,\mathrm{f}}^{\;2}/C_{22\,\mathrm{f}})}$$

$$= 3\,\sqrt[3]{G_{\mathrm{core}}^{\;2}}\left(\sqrt[3]{\tilde{D}_{11\,\mathrm{f1}} + K_{21\,\mathrm{f1}}^{\;2}/C_{22\,\mathrm{f1}}} + \sqrt[3]{\tilde{D}_{11\,\mathrm{f2}} + K_{21\,\mathrm{f2}}^{\;2}/C_{22\,\mathrm{f2}}}\right) . \qquad (66)$$

For sandwich panels with equal facings that are uniform in thickness direction the term $K_{12\,\mathrm{f}}$ vanishes while the facing bending stiffness applies to $D_{11\,\mathrm{f}} = t_{\mathrm{f}}^{3}\, E_{\mathrm{f}}/(12\,(1-\nu_{\mathrm{f}}^{2}))$.

Substitution of this expression for $D_{11\,\mathrm{f}}$ gives for the critical panel loading:

$$-N_{\mathrm{x\,cr}} = 2.621\; t_{\mathrm{f}}\;\sqrt[3]{G_{\mathrm{core}}^{\;2}\, E_{\mathrm{f}}/(1-\nu_{\mathrm{f}}^{2})} \; .$$

Imperfection sensitivity

For an isotropic-type of core for which the shear modulus is 1/2.6 times the Young's modulus ($\nu_{\mathrm{c}} = 0.3$), the solution derived here becomes: $-\sigma_{\mathrm{x\,cr}} = 0.953\,\sqrt[3]{G_{\mathrm{core}}\,E_{\mathrm{core}}\,E_{\mathrm{f}}/(1-\nu_{\mathrm{f}}^{2})}$.

In [6] the critical stress for face wrinkling of a sandwich plate with an isotropic core is reported as $-\sigma_{\mathrm{x\,cr}} = Q\,\sqrt[3]{G_{\mathrm{core}}\,E_{\mathrm{core}}\,E_{\mathrm{f}}/(1-\nu_{\mathrm{f}}^{2})}$ where the solution is reported in [6] as $Q = 0.825$. The larger factor $Q = 0.953$ in the solution presented here is because the elasticity of the core in thickness direction is not acounted.

To account for local imperfections of the facings [17] recommends the expression $-\sigma_{\mathrm{x\,cr}} = 0.5\,\sqrt[3]{G_{\mathrm{core}}\,E_{\mathrm{core}}\,E_{\mathrm{f}}}$ which is a reduction with a factor of about 0.57.
This reduction looks large but one has to realise that the sandwich facings in a blade loaded by bending may be not-equally loaded.

In the European BUCKBLADE research project, calculations with a non-linear FEM package for the sandwich tail panel buckling of a rotor blade showed a 0.8 times smaller critical load than calculations with a linear FEM package, see chapter 6 of [14], where the experimental bucking load of the sandwich tail panel was another factor 0.8 times smaller. These investigations and the fact that the failure mode of the experiment was visibly face wrinkling (near a small imperfection) learned that application of an imperfection sensitivity factor of 0.5 to 0.6 for face wrinkling is reasonable. For the same 'sandwich tail panel failure' the prediction for face wrinkling (at the location of experiment failure) was about 2 times higher than the experimental buckling load, even after reduction with a factor 0.6. The test specimen however showed a strong notch (local imperfection) in the outer facing at the location where the tail panel has collapsed.

Rewriting the expression for the critical stress to an expression for the critical panel load for symmetric sandwich panels (equal facings) and using a reduction factor of 0.5 gives:

$$-\,N_{\mathrm{x\,cr}} = 2.2\,\sqrt[3]{G_{\mathrm{core}}\,E_{\mathrm{core}}\,(\tilde{D}_{11\,\mathrm{f}} + K_{21\,\mathrm{f}}^{\;2}/C_{22\,\mathrm{f}})} \; . \qquad (67)$$

Notes

- The solution given in this section is an under-estimation for panels with finite width, because the deformation pattern is assumed prismatic.

- For facing laminates that are symmetric in thickness direction the terms $K_{21\,\mathrm{f}}$ are zero while the longitudinal bending stiffness is simply $D_{11\,\mathrm{f}}$.
- For panels with transverse curvature the term $K_{21\,\mathrm{f}}$ is small compared to $1/R_{\mathrm{y}}$.
- The expression for the support stiffness of the isotropic core is based on the assumption that the core is infinitely thick while the restraint for in-plane strain of the core near the facing is omitted. As a result the support stiffness of the core ($k_{\mathrm{support}} = 2\,G_{\mathrm{core}}\,\mu_{\mathrm{x}}$) is an under-estimation.

C PANEL AVERAGE PROPERTIES

Many of the more simple buckling load prediction methods such as the 'Design rules' in chapter 2 and the analytical solutions presented in Appendix B are formulated for panels with uniform material properties, uniform geometry, and a uniform or linear distribution of the longitudinal loading. These uniform properties can be calculated as straightforward average properties over the panel width (arithmetic averages). This however would give the same contribution of the local properties near the panel edges and those near the centre while near the edges the deformation is smaller. Also for the panel average load distribution the axial loading near the edges of the panel has a much smaller influence on the buckling load than the loading in the centre of the panel. Because of the different contributions of the edge areas and the central area of the panel, it was chosen to calculate the panel average stiffness properties with a weighting function that is based on the contribution of the stiffness to the deformation energy for the 'expected' (approximate) failure mode. This expected failure mode is considered for long panels with simply-supported edge constraints.

A similar approach is applied to calculate the panel average loading.

For flat and weak curved panels with uniform compressive loading and with the assumption of simply supported edge constraints, the expected failure mode has one half-wave in transverse direction. For strong curved panels, or panels with compression in only one edge the failure mode has a transverse half-wave dimension that is much smaller than the panel width. The latter 'effective half-wave width' is calculated following the description given in section 2.2.

C.1 Panel average stiffness terms

The analytical solution of long axially compressed plates with simply-supported edges is a failure mode with one half-wave ($\sin(\pi\, y/b)$) in transverse direction for the variable w, see section B.1. With the strain-displacement relations given in section A.3 one may find that the curvatures κ_{x} and κ_{y} and the strains ϵ_{x} and ϵ_{y} also have a $\sin(\pi\, y/b)$ function in transverse direction. Applying the $_{,y}$ derivatives in the linear terms of the expressions for the strains (32) one may derive that γ_{xy}, κ_{xy}, and v have a $\cos(\pi\, y/b)$ function in transverse direction.

Using these functions in the expression for the adjacent strain energy in the collapse mode for buckling of a panel (see (33) in section A.4) gives that the following matrix stiffness terms have a $(\sin(\pi\, y/b)\,)^2$ contribution to the energy expression: A_{11}, A_{12}, A_{22}, B_{11}, B_{12}, B_{22}, D_{11}, D_{12}, and D_{22}. These stiffness terms will also be averaged after multiplication with the weighting function $1 - \cos(2\,\pi\, y/b) = 2\,(\sin(\pi\, y/b)\,)^2$. For the panel average matrix-stiffness terms A_{66}, B_{66}, and D_{66} a $1 + \cos(2\,\pi\, y/b) = 2\,(\cos(\pi\, y/b)\,)^2$ weighting function is used.

The panels of most rotor blades are of orthotropic material, for which reason many of the design methods are formulated for orthotropic material properties only (except for the few anisotropy terms in the solution in section B.4). Most solutions for orthotropic panels are formulated in terms of the matrix $K = A^{-1}\,B$ and the 'reduced stiffness matrix' $\tilde{D} = D - B\,A^{-1}\,B$, while the 'anisotropy' terms B_{16}, B_{26}, D_{16}, and D_{26} are not accounted for. This means that $K_{66} = B_{66}/A_{66}$ and $\tilde{D}_{66} = D_{66} - B_{66}^2/A_{66}$ from which it can be derived that the panel average stiffness terms K_{66} and $\tilde{D}_{66}$ should also be integrated with the weighting function $1 + \cos(2\,\pi\, y/b)$. For orthotropic panels it follows similarly that K_{ij} and $\tilde{D}_{ij}$ with (*i* = 1, 2 and *j* = 1, 2) have a $(1 - \cos(2\,\pi\, y/b)\,)$ contribution to the adjacent deformation energy during collapse, so that their panel average values should be integrated using the weighting function $(1 - \cos(2\,\pi\, y/b)\,)$.

C.1.1 Panel average transverse bending stiffness

The panel average value of the transverse bending stiffness term $\tilde{D}_{22}$ is calculated on a slightly different basis as the other stiffness terms. For a collapse mode in which only the transverse deformation is considered, the transverse bending moment M_{y} will be a continuous function while the transverse curvature κ_{y} of the collapse mode will depend on the transverse bending moment and the local distribution of the stiffness $\tilde{D}_{22}$ by $\kappa_{\mathrm{y}} = M_{\mathrm{y}}/\tilde{D}_{22}$. Since the deformation energy is proportional with the product of κ_{y} and M_{y} one may conclude that for a continuous distribution of M_{y} the panel average transverse stiffness should be calculated by integration of its inverse value $1/\tilde{D}_{22}$. The weighting function will still be $(1 - \cos(2\,\pi\,y/b)\,)$.

Integration of $\tilde{D}_{22}$ will give a substantial contribution for (very) local high values, and is therefore un-conservative for buckling analyses. Integration of $1/\tilde{D}_{22}$ will give a substantial contribution for (very) local low values, and is therefore conservative.

C.1.2 Panel average anisotropy terms

In the prediction methods discussed here, only the analytical solution described in section B.4 ('method 2' in Farob) uses some of the anisotropy terms. Because the anisotropy terms describe interaction between loading in x or y direction and shear- or twist loading, their contribution to the deformation energy deals with bending (mainly in the centre of the panel) and shear (mainly near the panel edges). For this reason the panel average anisotropy terms C_{16} and C_{26} are calculated from straightforward averageing, so without a weighting function. These anisotropy terms are used in the extension of the panel method that is described in section B.2.

C.1.3 Panel average out-of-plane flexibilities

For integration of the panel average out-of-plane shear flexibilities one could use the fact that the analytical solution for uniform panels has a $\sin(\pi\,y/b)$ distribution for γ_{xz} and a $\cos(\pi\,y/b)$ distribution for γ_{yz}. However the contribution of out-of-plane shear flexibility (or deformation) can be described as reduction of the panel bending deformation energy. In the 'Design rules' the relation between bending stiffness and out-of-plane shear flexibility appears as a stiffness ratio: e.g. $r_1 = (\pi/b)^2\,(\sqrt{\tilde{D}_{11}\,\tilde{D}_{22}}\,/A_{55})\,(1+\beta)/2$ and $r_2 = (\pi/b)^2\,(\tilde{D}_{22}\,/A_{44})\,(1+\beta)/2$.

The longitudinal out-of-plane shear flexibility A_{55}^{-1} of the collapse mode is still assumed to have a $\sin(\pi\,y/b)$ function in transverse direction. The deformation energy in longitudinal direction is proportional with the longitudinal bending stiffness $\tilde{D}_{11}$. This deformation energy is reduced by the longitudinal out-of-plane shear flexibility from which one can deduce that the panel average longitudinal out-of-plane shear flexibility A_{55}^{-1} should be integrated with the weighting function: $\tilde{D}_{11}\,(1 - \cos(2\,\pi\,y/b)\,)$ and finally divided by the panel average value of $\tilde{D}_{11}$.

The transverse out-of-plane shear deformation is proportional with the transverse out-of-plane panel loading Q_{yz}. Knowing that for the collapse mode the load distributions between the panel edges are continuous functions of y one may conclude that the contribution of the transverse out-of-plane shear flexibility A_{44}^{-1} is not directly related to e.g. the transverse bending stiffness $\tilde{D}_{22}$. So the panel average value of A_{44}^{-1} is integrated simply with the weighting function $(1 + \cos(2\,\pi\,y/b)\,)$.

The approach to calculated the panel average out-of-plane shear flexibilities is based on the assumption that transverse bending and longitudinal bending are independent from each other. The existence of panel twisting moments M_{xy} in the equations (37) and (38) show that this approach is not fully complete. Finding a more realistic way to calculate panel average stiffness properties requires more detailed knowledge about the collapse mode so that it in fact requires solution of the collapse mode of non-uniform panels, such as presented in section A.4 - A.5.

C.2 Panel average loading

The integration of the panel average loading is based on the contribution to the released strain-energy for the collapse mode. Also here the collapse mode is assumed to have a $\sin(\pi\, y/b)$ function in transverse direction. Here the axial load distribution $N_{\mathrm{x}\,0}$ for each element in the panel is calculated from the strains in the nodes and the in-plane flexibility matrix C:

$N_{\mathrm{x}\,0} = (\epsilon_{\mathrm{x}\,0} - C_{12}\, N_{\mathrm{y}\,0}\; - C_{16}\, N_{\mathrm{xy}\,0})/C_{11}$.

This expression is based on the fact that for a panel of a long thin-walled beam the shear loading $N_{\mathrm{xy}\,0}$ (and transverse loading $N_{\mathrm{y}\,0}$) are prescribed from equilibrium within the beam cross-section while the axial strain directly follows from bending of the blade as beam. Note that the strains and loads are negative in compression. Also note that for orthotropic panels the flexibility term C_{16} is zero for which the axial loading does not have any interaction with the shear loading.

The resulting distribution of $N_{\mathrm{x}\,0}$ is used to integrate the panel average axial load distribution with a weighting function $(1 - \cos(2\,\pi\, y/b)\,) \;= 2\,(\sin(\pi\, y/b)\,)^2$.

The panel average loading is used among others in the 'Design rules' that are based on the expressions for linear panel load distribution over the transverse coordinate. For this purpose the panel average in-plane bending moment is calculated from the axial load distribution $N_{\mathrm{x}\,0}$. The weighting function now has an additional linear term, and thus becomes $(2\,y/b - 1)\,(1 - \cos(2\,\pi\, y/b)\,)$. After integration, the panel average bending moment is scaled such that it would fit the loading $N_{\mathrm{x}\,0}$ in the panel edges for pure in-plane bending of a uniform plate.

C.3 Assumed collapse mode for strong curved panels

For panels with a strong transverse curvature $1/R_{\mathrm{y}}$ the collapse mode may have multiple half-waves in transverse direction. For a panel with uniform material, curvature, and axial loading the collapse mode has a $\sin(n\,\pi\, y/b)$ shape with n the number of transverse half-waves. The number of half-waves n of the most critical collapse mode increases with increasing panel width and curvature, and is evaluated with the expressions of Van der Neut [6] from the curvature parameter Z. Following these expressions, the critical load applies to the form of slightly curved panels for $\eta\, Z/\pi^2 \;\leq\; k_{\mathrm{c\,flat}}$, see also (4) in section 2.2. From this condition it is concluded that the here-called "critical curvature" applies to $\eta\, Z_{\mathrm{crit}}/\pi^2 = k_{\mathrm{c\,flat}}$.

If the panel curvature parameter Z is smaller than $(1.25)^2$ times the critical curvature Z_{crit} it is assumed that the collapse mode has 1 half-wave in transverse direction and the panel average loading and material properties are integrated with the weighting functions as described in the previous sections. If the panel curvature Z is larger than $(1.25)^2$ times the critical curvature the critical half-wave width is calculated with $b_{\mathrm{crit}} = b\,\sqrt{Z_{\mathrm{crit}}/Z}$, see section 2.2.

The collapse mode of panels with uniform curvature, material properties, and loading has a number of n half waves of equal size (and shape). For those panels it would be appropriate to use $2\,(\sin(n\,\pi\, y/b)\,)^2$ weighting functions. However, for non-uniform material and loading the collapse mode will not have a regular periodic sinusoidal shape although near the panel edges the shape may have a sinusoidal character. The weighting functions that are thus applied to calculate the panel average stiffness properties with subscripts 1 1, 1 2, 2 1, and 2 2 are described by a $(b/(2\,b - b_{\mathrm{crit}}))\,(1 - \cos(\pi\,(y - y_{\mathrm{edge}})/b_{\mathrm{crit}})\,)$ function over the panel-area near the edges with width $b_{\mathrm{crit}}/2$. In the panel area in between the weighting function is simply the constant factor $(b/(2\,b - b_{\mathrm{crit}}))$. This factor is such that the panel average value of the weighting function is 1.

These weighting functions are also used to integrate the panel average loads.

The weighting functions applied to calculate the panel average stiffness properties with subscripts 6 6 are simply 2 minus the latter weighting functions.

C.4 Panel average curvature

In particular the leading edge panels of a rotor blade cross section with relative thin airfoils may have a strongly non-linear curvature. For application of the 'Design rules' based on the analytical solution presented in section B.1 the panel geometry has to be expressed in one representative 'panel average' curvature.

For the panel average curvature it was chosen to use the curvature of an arc that is drawn through the edge-points and a point in the centre of the panel. For this 'centre point' one faces the choice between e.g. the 'arc-length' middle of the contour and the point on the contour that has the same distance from each of the edge-points. For panels with a uniform or symmetric curvature the use of both these 'centre point' definitions gives the same circular arc. For panels with non-uniform (asymmetric) curvature, it was concluded that the latter definition gives a smaller panel average curvature and is therefore conservative. Based on the fact that an asymmetric curved panel tends to fail by buckling near the 'most flat' side, the latter –more conservative– 'centre point' was used to calculate the panel average curvature.

After calculating the curvature through the 3 points the panel width used is simply the arc-width of the real non-uniform curvature.

C.4.1 Average curvature for S-shaped panels

For panels with an S-shape such that the centre point is linear between the edge points, the calculated panel average curvature is (nearly) zero. This occurs for tail-panels at the aerodynamic pressure side of rotor blades. For these panels it may happen that the calculated collapse load is far too large. In practice the S-shaped curvature also contributes to the buckling resistance. Including the geometric contribution of the S-shaped panel geometry to the buckling load would be a major improvement of accuracy of the predictions with the 'Design rules' (method = 1) and of the analytical solution (method 2).

D DESCRIPTION OF PANEL-BASED BUCKLING ROUTINES

Part of the work within the BLADKNIK project was addressed to the development of panel-based routines that are fast compared to the program Finstrip but also may be slightly less accurate because they do not include the complete structural integrity of a cross section. The fact that these tools are fast allows many buckling load analyses such as for the time-series of load combinations from design analyses or for variations of structural design. These two applications require that the routines should be accurate in the modelling of material properties and also for non-uniform and combined loading.

Within this scope the following FORTRAN routines were developed:

panini2 Routine that calculates the panel stiffness matrix elements of a given layup. These stiffnesses are used by the buckling-load prediction routines *bucpan3* and *bucweb2* and also by subroutine *rlamod3*.

bucpan3 For buckling analysis of the outer contour, that may have varying geometry and material layup but of which the panel loading is dominated by longitudinal compression.
This routine returns the critical load factor following different so-called 'methods' in the Farob input. The algorithm for 'method 1' is based on the 'Design Rules' for curved sandwich panels described in section 2.6.
For other (higher order) 'methods' routine *bucpan3* has a loop that searches over the half-wave length with the smallest buckling load factor. For the buckling load factor as function of half-wave length, routine *bucpan3* has a call for routine *rlamod3*.

bucweb2 For buckling analysis of the shear web, of which the geometry is assumed (nearly) flat but of which the loading can be complex. The algorithm is based on the 'Design rules' for (nearly) flat sandwich panels with various loading given in section 2.7.

rlamod3 This routine returns the load factor for a curved orthotropic sandwich panel with a given half-wave length. For an input argument 'imethd' equal to 2 this routine returns the analytical solution for panels with uniform curvature and material properties.
For larger values of the input argument 'imethd' this routine returns the numerical solution for a panel with non-uniform curvature and non-uniform layup.

The call for the routines *bucpan3* and *bucweb2* is identical. All these routines use panel stiffness properties that are calculated in routine *panini2*. These routines are used by the program Farob under the design package Focus. Routine *panini2* is also used in the tool Crostab for the calculation of the sectional stiffnesses from the laminate stiffness properties.

A description of the parameter list and of the algorithm of each of these routines is given in the following sections.

D.1 Initialisation routine 'panini2'

Routine *panini2* calculates the elements of the stiffness matrices for a given 'stacking' of layers. The location of the layers with respect to the reference plane is on the negative z side if the layer-thickness is negative, and on the positive z side if the layer thickness is positive. The A, B, and D stiffness matrices are calculated with respect to the $z = 0$ plane, which usually is the outer contour for a rotor blade panel. The reason to chose the $z = 0$ plane instead of e.g. the 'neutral plane' for longitudinal tensile forces in the panel is that the routine with a 'rigorous solution' (see section A.4 - A.5) accounts for the variations in material layup along the panel width.

Argument list of 'panini2'

The arguments of *panini2* are described here on basis of their FORTRAN variable name:

Name	Type	I/O	Description
imesg	INTEGER	In	Unit number for writing messages.
istck	INTEGER	In	Number of the stacking for which stiffnesses are calculated.
inlay	INTEGER	In	Number of layers in this stacking.
thlay	REAL(*)	In	Array with the thicknesses [L] of each layer.
filay	REAL(*)	In	Array with orientations [rad] of each layer.
imlay	INTEGER(*)	In	Array with material indices of each layer.
e1mat	REAL(*)	In	Array with longitudinal stiffnesses of each material.
e2mat	REAL(*)	In	Array with transverse stiffnesses of each material.
rnumat	REAL(*)	In	Array with Poisson's ratios of each material.
g12mat	REAL(*)	In	Array with in-plane shear moduli of each material.
g13inv	REAL(*)	In	Array with out-of-plane shear flexib. $x-z$ of each material.
g23inv	REAL(*)	In	Array with out-of-plane shear flexib. $y-z$ of each material.
c11	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
c12	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
c22	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
c16	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
c26	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
c66	REAL	Out	Element of the in-plane flexibility matrix; $C = A^{-1}$.
zref	REAL	Out	z-location [L] of the reference plane for longit. stiffness.
lerror	LOGICAL	Out	If 'TRUE' an error has occurred.

The elements 'cij' of the in-plane flexibility matrix C and the z-location of the reference plane for longitudinal stiffness are only used by Crostab. The length-dimensions (here denoted with [L]) can be [m] or [mm] or any other unit, as long as this is done consistently for all input and output arguments of *panini2*, *bucpan3*, and *bucweb2*.

Result

The result of calling routine *panini2* are the elements of the stiffness matrices A, B, and D, which are calculated with respect to the $z = 0$ plane. From these matrices the matrix $K = A^{-1} \cdot B$ and the 'reduced stiffness matrix' $\tilde{D} = D - B \cdot A^{-1} \cdot B$ are calculated. The use of the matrices $C = A^{-1}$, K, and $\tilde{D}$ is introduced by expressing the in-plane strains in the curvatures and the in-plane panel loads, see section A.6 and section B.1.

For the purpose of the numerical solution for non-uniform panels also the stiffness properties S_{11}, S_{12}, S_{21}, S_{22}, T_{11}, T_{12}, and T_{22} (see section A.4) are assigned in *panini2*.

All elements of the matrices C, B, K, D, $\tilde{D}$, S, T, the properties A_{22} and A_{66}, and the sandwich shear flexibilities (A_{55}^{-1}) and (A_{44}^{-1}) are stored in include file 'PANDAT2.FI' using the index 'istck' of the stacking for which *panini2* is called.

D.2 Routines 'bucpan3' and 'bucweb2'

Routines *bucpan3* and *bucweb2* are called by Farob. The call and the argument list of these routines are identical, although Farob calls *bucweb2* for buckling of the shear webs and *bucpan3* for buckling of the contour panels. The algorithms for the buckling load factor in routine *bucpan3* and *bucweb2* are based on the 'Design rules' that are described in chapter 2. For routine *bucweb2* the panel curvature is assumed to be very small (because shear-webs are flat) so that branches for strong curvature are omitted.

Routine *bucpan3* also offers more detailed buckling load analyses, such as the analytical solution in section B.1 - B.3, or the numerical solution for non-uniform panels in section A.4 - A.5. For the latter solutions routine *bucpan3* has a call for subroutine *rlamod3*, see section D.3. Routine *rlamod3* is called depending on the value of input argument 'imethd'.

Argument list of 'bucpan3' and 'bucweb2'

Following is a description of the arguments of routine *bucpan3* and *bucweb2* on basis of their FORTRAN variable name:

Name	Type	I/O	Description
imesg	INTEGER	In	Unit number for writing messages.
inpnts	INTEGER	In	Number of coordinate points.
xcoord	REAL(*)	In	Array with x coordinates [L] of the panel geometry.
ycoord	REAL(*)	In	Array with y coordinates [L] of the panel geometry.
dels	REAL(*)	In	Distance [L] between coordinate points '(ip)' and '(ip+1)'.
istack	INTEGER(*)	In	Array with indices for the material stacking. Here 'istack(ip)' refers to the stacking between coordinates with index '(ip)' and '(ip+1)'.
strain	REAL(*)	In	Array with longitudinal strains in point ('xcoord','ycoord').
qflow	REAL(*)	In	Array with shear loading [N/L] between '(ip)' and '(ip+1)'.
rnyav	REAL	In	Panel-average transverse loading [N/L], tension positive.
curvx	REAL	In	Longit. blade curvature [1/L] toward flapwise (x) direction.
curvy	REAL	In	Longit. blade curvature [1/L] toward lagwise (y) direction.
twist	REAL	In	Geometric panel twist in longitudinal direction [1/L].
rlemin	REAL	In	Minimum length [L] of the collapse mode.
rlemax	REAL	In	Maximum length [L] of the collapse mode.
imethd	INTEGER	In	Indicates how detailed the buckling analysis should be.
lshape	LOGICAL	In	If 'TRUE' also return the collapse mode ('dx','dy').
lfree	LOGICAL	In	If 'TRUE' all buckling modes with a half-wave length between 'rlemin' and 'rlemax' have to be examined.
dx	REAL(*)	Out	x coordinate of the (dimensionless) collapse mode.
dy	REAL(*)	Out	y coordinate of the (dimensionless) collapse mode.
rlacr	REAL	Out	Factor on the applied load for which buckling is calculated.
rlencr	REAL	Out	Half-wave length [L] for the most critical collapse mode.
thetcr	REAL	Out	Orientation [rad] of the waves of the collapse mode.

The longitudinal strains 'strain(ip)' are defined at the coordinate points 'xcoord(ip)', 'ycoord(ip)'. The shear loading 'qflow(ip)' is defined as constant over an element between coordinate points with index '(ip)' and '(ip+1)'.

If routines *bucpan3* and *bucweb2* are called with input argument 'imethd' equal to 1, the solution for the buckling load factor 'rlacr' is calculated with the 'Design rules' (chapter 2) for which the minimum and maximum half-wave length (input arguments 'rlemin' and 'rlemax') and the flag 'lfree' for using a fixed or free length are not used. This is because the 'Design rules' are formulated for very long panels.

Panel-average geometry parameters

In both routines *bucpan3* and *bucweb2* the 'panel average' geometry parameters are calculated so that the 'Design rules' and the analytical solutions from Appendix B can be applied. These geometry parameters include the panel width and the panel curvature, of which the panel width is simply the integrated length along the geometric coordinates (xcoord,ycoord).

The approach to calculate the panel average curvature is described in section C.4. The panel curvature is defined positive if the concave side is toward the z-axis with which the stacking is given. This z-axis direction is on the 'right' side following the (xcoord,ycoord) coordinate points of the panel. For the conventional definition of a rotor blade in Farob the positive z direction is toward the inner-side of the blade contour, so that the panels near the leading-edge will have a positive curvature.

Result

The result of routines *bucpan3* and *bucweb2* is the factor 'rlacr' on the specified loading (arrays 'strain', 'qflow', and 'rnyav') for which bifurcation buckling is calculated. In addition to this factor, also the half-wave length and the orientation of the 'buckles' w.r.t. the transverse direction are assigned to the variables 'rlencr' and 'thetcr'.

Type of analysis based on 'imethd'

The calculation of the buckling load factor 'rlacr' by *bucpan3* and by *bucweb2* partly depends on the value of the input argument 'imethd':

0 Assign the buckling load factor for non-sandwich panels loaded by linear axial compression only (*bucpan3*) or shear loading and in-plane bending only (*bucweb2*);

1 Use the Design rules for sandwich panels, based on the load-interaction rules from NEN6771 for the estimated critical half-wave width following the solution of Van der Neut;

2 + (*bucweb2*) Return the Design rules -solution including transverse loading, see section 2.7;

2 (*bucpan3*) Return the analytical solution for uniform panels, see section B.1 - B.3;

3 + (*bucpan3*) Return the numerical solution for non-uniform panels, see section A.4 - A.5.

For values of 'imethd' up to 1, the maximum half-wave length of the deformation pattern and the flag 'lfree' for a free/fixed maximum length are not used in routine *bucpan3*. In routine *bucweb2* the parameters for the allowable half-wave length are not used at all.

If the solution of the load-factor is negative, the loading is assumed not buckling-critical (e.g. dominantly tension) for which the output parameter 'rlacr' is set to the "very large" value 2.E+30. In that case the output value of 'rlencr' is set to zero.

Output of the collapse mode

If routines *bucpan3* or *bucweb2* are called with input argument 'imethd' not larger than 1 (Design rules) and the value of 'rlacr' appears positive but smaller than 2.E+30 then the output value of the half-wave length of the collapse mode '*rlencr*' is set to the value for the most critical collapse mode of orthotropic sandwich plates, such as derived in section B.4.

For *bucpan3* and *bucweb2* the output argument for the orientation of the collapse mode '*thetcr*' is a linear combination of that for compression (which is zero) and the value for shear loading, depending on the relative fraction of shear loading. If however the applied loading is dominated by transverse compression the angle '*thetcr*' is set to $\pi/2$.

If 'lshape' has the value '.FALSE.' or if no positive critical load factor is found then the collapse mode is set to the undeformed state (the pairs of variables '*dx*','*dy*' are set to zero).

For routine *bucpan3* the collapse mode depends on the type of solution 'imethd', provided that 'lshape' has the value '.TRUE.'.

0 a number of sinusoidal half-waves, that fits best to the value of the 'critical half-wave lengt' b_{crit}, see section 2.2. The deformation is assigned perpendicular to a curved line between the panel edges.

1 a number of sinusoidal half-waves that fits best to the value of the 'critical half-wave length' b_{crit}, see section 2.2. For *bucweb2* the collapse mode is assigned perpendicular to a (curved-) line between the panel edges. For *bucpan3* the collapse mode is assigned accounting for the geometric non-linear combination with panel curvature.

2 the number of sinusoidal transverse half-waves from the analytical solution of *rlamod3*. In fact the collapse mode is assigned in routine *rlamod3*, that finally is (or should be) called with a value '.TRUE.' for 'lshape'. This collapse mode is assigned as perpendicular to a uniform curved line between the panel edges.

3 + the collapse mode is the numerical solution for panels with non-uniform loading, curvature, and material properties.

Because the result of *bucpan3* and *bucweb2* is always an eigenvalue solution, the collapse mode is scaled to a maximum out-of-plane deformation of 1.0.

For a linear varying axial compressive loading, the collapse mode is multiplied with a decreasing function such that the least compressive part of the panel (or tensile part) has the smallest amplitude of the buckles.

D.3 Routine 'rlamod3'

Routine *rlamod3* returns the solution of the load factor for which bifurcation buckling is calculated for a specified half-wave length. Under Farob, routine *rlamod3* is called by routine *bucpan3* with which this routine was developed.

Argument list of 'rlamod3'

Following is a description of the arguments of routine *rlamod3* on basis of their FORTRAN variable name:

Name	Type	I/O	Description
imesg	INTEGER	In	Unit number for writing messages.
inpnts	INTEGER	In	Number of coordinate points.
xcoord	REAL(*)	In	Array with x coordinates [L] of the contour.
ycoord	REAL(*)	In	Array with y coordinates [L] of the contour.
dels	REAL(*)	In	Distance [L] between coordinate points '(ip)' and '(ip+1)'.
istack	INTEGER(*)	In	Array with indices for the material stacking.
			Here 'istack(ip)' refers to the stacking between
			coordinates with index '(ip)' and '(ip+1)'.
strain	REAL(*)	In	Array with longitudinal strains in point ('xcoord','ycoord').
qflow	REAL(*)	In	Array with shear loading between '(ip)' and '(ip+1)'.
rnyav	REAL	In	Panel-average transverse loading [N/L], tension positive.
width	REAL	In	Integrated panel-width [L].
curvp	REAL	In	Panel-average transverse curvature [1/L].
curvx	REAL	In	Longit. blade curvature [1/L] toward flapwise direction.
curvy	REAL	In	Longit. blade curvature [1/L] toward edgewise direction.
twist	REAL	In	Geometric panel twist in longitudinal direction [1/L].
rlengt	REAL	In	Half-wave length [L] for which routine *rlamod3* is called.
imethd	INTEGER	In	Indicates how detailed the buckling analysis should be.
lfirst	LOGICAL	In	If 'TRUE' the panel-average loads and stiffnesses are assigned.
lcurv	LOGICAL	In	If 'TRUE' the longitudinal curvatures are scaled with 'rlacr'.
lshape	LOGICAL	In	If 'TRUE' also assign the collapse mode (dx,dy).
rlacr	REAL	In	Lower bound (start value) of the buckling load factor.
		Out	Factor on the applied load for which buckling is calculated.
dx	REAL(*)	Out	x coordinate of the (dimensionless) collapse mode.
dy	REAL(*)	Out	y coordinate of the (dimensionless) collapse mode.
thetcr	REAL	In	Fraction of the critical half-wave width versus panel width.
		Out	Orientation [rad] of the waves of the collapse mode.

The coordinates 'xcoord' and 'ycoord' and the strains in the elements are identical to the similar arguments of routine *bucpan3*.

Analytical solution for uniform panels

The algorithm that is used to calculate the critical load factor depends on the value of the input argument 'imethd'. For 'imethd' not larger than 2, routine *rlamod3* returns the analytical solution for uniform double curved sandwich panels, see section B.1 through B.3, using panel average material properties that are calculated with the weighting functions reported in Appendix C. In this solution the longitudinal and transverse curvature are taken into account. For curved panels this longitudinal curvature is finally scaled down by multiplication with (distance between panel edges)/(arc-length width). The latter implies that for a panel with a semi-circular cross section the longitudinal curvature used for the analysis is thus $2/\pi$ of the blade curvature.

If the value of the input argument 'lcurv' is `.TRUE.` the influence of the longitudinal curvature is scaled with the load factor 'rlacr'.

The non-uniform axial loading is represented with a constant (compressive) loading and an in-plane bending-moment. The contribution of the in-plane bending moment to the axial compressive panel loading is taken into account using the rules given in NEN6771 [19].

To save CPU time the panel average material properties and the panel average loading are only calculated if 'lfirst' has the value `.TRUE.` and are saved for further calls. In routine *bucpan3* the value of 'lfirst' is set `.TRUE.` before the first call of *rlamod3*, and is set .FALSE. for further calls.

Numerical solution for non-uniform panels

If routine *rlamod3* is called with a value of 'imethd' equal 3 (or higher) the load factor is obtained from the solution of the 2-point boundary value problem of which the equations are derived in section A.4 - A.5. This solution is obtained by incrementing the load factor until the boundary conditions on both ends of the panel are satisfied, which indicates a collapse mode (bifurcation point). In this process the initial value for the incrementing load factor is some fraction of the load factor from the analytical solution similar as for 'imethd 2' calculated without the influence of transverse compressive loading and without the anisotropy terms C_{16} and C_{26} .